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THE NORMAL MODE APPROACH TO THE SOLUTION OF THE INITIAL VALUE PROBLEM FOR PLASMA OSCILLATIONS

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P. E. Bolduc

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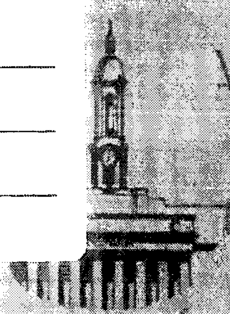

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on

"The Normal Mode Approach to the Solution of the Initial Value Problem for Plasma Oscillations"

by

P. E. Bolduc

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Ionosphere Research Laboratory

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TABLE OF CONTENTS

Abstract

10170

The first order determination of the properties of a plasma immersed in a constant, external, magnetic field is carried out. Normal mode analysis is utilized in an initial value problem based on plane symmetry. Methods for determining expansion coefficients for the transverse and longitudinal modes, separately, are deduced.

AUTHOR

I. INTRODUCTION

1.1 Early Description

The history of plasma physics can be traced back to Langmuir's gas discharge experiments of the late 1920's. He discovered that some electrons in the discharge had energies larger than the potential difference across the tube. The explanation for the above and associated phenomena appeared in a theoretical paper by Tonks and Langmuir in that same year [Tonks and Languir, 1929]. They postulated that the discharge not only ionized the gas in the tube but that it also caused some spatial rearrangement of the generated electron concentration thus creating strong local electric fields. This would explain the energy source of the high velocity electrons observed. They then carried their analysis one step further by predicting that the system's attempt to regain over-all neutrality would result in electron or plasma oscillations, which in the absence of collisions and of thermal motion (a zero temperature or cold plasma model was chosen) would execute their periodic motion at the derived characteristic frequency, $w = w_p$, ad infinitum. In terms of more basic parameters, w_p was shown to be given by

$$(1.1) \quad w_p = \sqrt{\frac{4\pi n_o e^2}{m}}$$

where w_p is the plasma frequency and n_o, e, m are the electronic concentration, charge, and mass, respectively.

One ad hoc postulate was added to their theory. They supposed that the transverse oscillations could also exist and that these would account for the radiation observed to linger after the main discharge had subsided. Spectroscopic analysis has indicated, however, that this radiation is due to Bremsstrahlung and to the quantized radiative processes that occur in the ion-electron recombination [Griem, 1961].

The parameter w_p is of primary importance in all problems dealing with the interaction of electromagnetic radiation with ionized media. It may be shown [Jackson, 1962] that the medium's index of refraction, n, is given by

$$(1.2) \quad n = \sqrt{1 - \frac{w_p^2}{w^2}}$$

where w is the frequency of the propagating electromagnetic wave.

The above relation implies that transmission obtains when $w > w_p$ and that attenuation and reflection result when $w < w_p$. The importance of Tonks and Langmuir's dynamical approach to this characteristic frequency lies on the fact it sheds some light on the mechanical nature of this periodic process.

1.2 Non-Equilibrium Statistical Mechanics

Two decades later, Vlasov [Vlasov, 1945] proposed that the problems of ionized gas physics could be more adequately treated by the methods and concepts of classical statistical mechanics. He showed that the Liouville equation leads to the familiar transport equation

$$(1.3) \qquad \frac{\partial F(\bar{r},\bar{u},t)}{\partial t} + \bar{u}\cdot\bar{\nabla}_r F + \bar{a}\cdot\bar{\nabla}_u F = \left(\frac{\delta F}{\delta t}\right)_{coll.}$$

where $F(\bar{r},\bar{u},t)$ is the probability of finding an electron in dr^3 centered at $\bar{r}$, and in du^3 centered at $\bar{u}$ at time, t; $\bar{a}$ is a smooth acceleration suffered by an electron due to a smeared out potential created by the presence of many other charged particles in its vicinity; $\left(\frac{\delta F}{\delta t}\right)_{coll.}$ is a violent redistribution of the electron's point in six dimensional configuration-velocity space brought about by a close binary collision either between an electron and an ion or an electron with a second electron.

Vlasov then proposed that the solution of the above equation coupled to that of the associated four Maxwell relations would constitute a complete description of the plasma state. It is from this point that much of the subsequent work in the quantitative description of plasma phenomena proceeds. In this work, however, we shall not concern ourselves with the validity of this singlet density transport equation but rather assume the latter and demonstrate its solution.

It is important to note here that the system is composed only of electrons, positive ions under various degrees of ionization, and of neutral gas atoms and / or molecules. Although all the particles of the system obey a transport equation, we shall assume that the distribution function for all particles except electrons is constant. We may readily justify this simplification by remembering that all other particles are at least 1836 times more massive than the electron. It is for this reason that we may speak of a uniform ion background which is such that the system is in an overall neutral state.

1.3 Microscopic and Macroscopic Descriptions

There exist two major approaches to the solution of the above mentioned set of five coupled partial differential equations. The first consists in the linearization of the transport or Boltzmann equation and the subsequent solution of the set of equations via transform techniques, partial differential equations (Ansatz solution), or the method of normal modes. This approach yields the value of all Maxwell field quantities and permits us to see how the plasma's distribution function, F, varies as a function of $\bar{r}$, $\bar{u}$, and t. This first method is usually referred to as the mirco-scopic description. The second approach consists in multiplying the linearized transport equation by 1, $m\bar{u}$ and $\frac{m\bar{u}\cdot\bar{u}}{2}$, respectively, and integrating the results over all velocity components. This yields the continuity, momentum transfer, and energy transfer equations, respectively, and together with the four Maxwell relations constitute the so-called macroscopic description of plasma parameters. This latter approach is usually called magneto-hydrodynamics. In the following work, we shall be concerned only with the former of these two methods.

1.4 Purpose

In this work, we shall show that the normal mode approach to the solution of the transport equation which was first clearly expounded by K. M. Case [Case, 1959] and later extended by Shure [Shure, 1962] can after some modification of the latter's work be applied to the solution of the initial value problem for the case of a temperature dependent plasma immersed in a constant, outside applied magnetic field. Furthermore, the relationship between the two approaches will be demonstrated and a particular initial value problem will be considered in some detail.

In order to enable ourselves to compare results with those of Landau and to outline the deficiencies in other approaches, we have felt it useful to include a brief description of these other techniques. These will include the transform, Ansatz and stationary phase approaches.

1.5 Transform Techniques

The first successful attempt at the solution of the transport equation for the case of a temperature dependent gas is attributed to L. D. Landau [Landau, 1946]. He showed that the linearized Boltzmann equation for the case of no collisions and where the only force term was due to the strong local electric fields could, in principle, be solved by transform methods. His linearized equation was

$$(1.4)\qquad \frac{\partial f(\bar{r},\bar{u},t)}{\partial t} + \bar{u}\cdot\bar{\nabla}_r f - \frac{e}{m}\bar{\nabla}_r\varphi\cdot\bar{\nabla}_u(n_oF_o) = 0$$

where he assumed that

$$(1.5)\qquad F(\bar{r},\bar{u},t) = n_oF_o(\bar{u}) + f(\bar{r},\bar{u},t)$$

$(r_o \rightarrow n_o)$

$$(1.6)\qquad n_oF_o \gg f(\bar{r},\bar{u},t)$$

$$(1.7)\qquad \bar{\nabla}_r\cdot\bar{\nabla}_r\varphi(\bar{r},t) = -4\pi e\int_{-\infty}^{\infty} f(\bar{r},\bar{u},t)\,du^3.$$

A Fourier transform of the space variable and a Laplace transform of the time lead to the transform equations

$$(1.8) \qquad f_{p,k} = \frac{1}{p + iku_z}\left[g_k(\bar{u}) + ik\frac{e}{m}n_o\varphi_{p,k}\,\frac{\partial F_o(|\bar{u}|)}{\partial u_z}\right]$$

and

$$(1.9) \qquad \varphi_{p,k} = \frac{4\pi e}{k^2}\,\frac{\displaystyle\int_{-\infty}^{\infty}\frac{g_k(u_z)}{(p+iku_z)}\,du_z}{\left[1 - \frac{4\pi ie^2}{km}n_o\displaystyle\int_{-\infty}^{\infty}\frac{dF_o}{du_z}\,\frac{du_z}{(p+iku_z)}\right]}$$

where

$$(1.10) \qquad g_k(u_z) = f_k(u_z,0).$$

Taking the inverse transforms of these functions then completes the solution and yields $f(z,\bar{u},t)$ and $\varphi(z,t)$ for any time, $t \geq 0$.

Landau's major contribution came as a result of an investigation of the poles in the expression for $\varphi_{p,k}$. The solution to the equation

$$(1.11) \qquad 1 - \frac{4\pi ie^2}{km}n_o\int_{-\infty}^{\infty}\frac{dF_o}{du_z}\,\frac{du_z}{(p+iku_z)} = 0$$

leads to the dispersion relation which bears his name:

$$(1.12) \qquad w = w_p\left[1+\frac{3}{2}k^2\lambda_D^2 - i\sqrt{\frac{\pi}{8}}\,\frac{1}{(k\lambda_D)^3}\,e^{\frac{1}{-2(k\lambda_D)^2}}\right]$$

where

$$(1.13) \quad \lambda_D \equiv \sqrt{\frac{KT}{4\pi n_o e^2}} \quad , \text{ Debye length,}$$

$k \lambda_D \ll 1$; this is the approximation condition used to derive the dispersion relation. The Debye length is the effective range of interaction between two charged particles. All charged particles at a distance greater than λ_D from a particular electron form a neutral background and therefore don't exert forces on it. Finally, we see that, in the limit of a cold plasma, Landau's results reduce to those predicted by Tonks and Langmuir, namely,

$$w \rightarrow w_p.$$

1.6 Ansatz Solutions

Although a strictly chronological development of the subject would dictate that we consider the Bohm and Gross contribution at this point, [Bohm and Gross, 1949], a far greater insight into the mathematical intricacies and their possible pit-falls can be obtained by next outlining the main results of a paper written by Berz [Berz, 1956] . He was able to show that the "Ansatz" [1]

[1] There seems to exist some confusion in terminology here. The German equivalent to substitution is Ersatz whereas Ansatz means a " start".

or substitution method first employed by Vlasov and later by Bohm and Gross leads to particular solutions of the transport equation and to an incorrect dispersion relation. The "Ansatz" method consists in the a priori assumption that all quantities having $\bar{r}$ and t as arguments vary as $\exp i(\bar{k}\cdot\bar{r} - wt)$.

Berz assumed variations of the distribution function in one direction only and linearized the transport equation as did Landau, i.e.

$$(1.14)\quad \frac{\partial f}{\partial t}(z,u_z,t) + u_z\frac{\partial f}{\partial z} + \frac{e}{m}E(z,t)\,\frac{\partial(n_o F_o)}{\partial u_z} = 0$$

where

$$(1.15)\quad \frac{\partial E}{\partial z}(z,t) = 4\pi e \int_{-\infty}^{\infty} f(z,u_z,t)\,du_z .$$

Assuming that

$$(1.16)\quad E(z,t) = E_o\, e^{i(kz - wt)}$$

but,

$$(1.17)\quad f(z,u_z,t) = f_k(u_z,t)\, e^{ikz}$$

and inserting into the above gives

$$(1.18)\quad \frac{\partial f_k(u_z,t)}{\partial t} + iku_z f_k + \frac{en_o E_o}{m}\, e^{-iwt}\, \frac{dF_o}{du_z} = 0$$

and

$$(1.19) \quad ikE_o e^{-iwt} = 4\pi e \int_{-\infty}^{\infty} f_k(u_z,t)\, du_z \; .$$

The solution of the transformed Boltzmann equation can be obtained by ordinary means and is

$$(1.20) \quad f_k(u_z,t) = i\frac{e}{m}\left[n_o E_o \frac{\frac{dF_o}{du_z} e^{-iwt}}{(ku_z - w)} + C(u_z)e^{-iku_z t}\right]$$

where $C(u_z)$ is the integration constant of the equation. Its specific form as a function of velocity can be obtained only by imposing the initial conditions.

Inserting the above result into equation (1.19) gives us the dispersion relation:

$$(1.21) \quad \left(\frac{k}{w_p}\right)^2 = \int_{-\infty}^{\infty} \frac{\frac{dF_o}{du_z}\, du_z}{(u_z - w/k)} + \frac{k}{E_o n_o} e^{iwt} \int_{-\infty}^{\infty} C(u_z)e^{-iku_z t}\, du_z .$$

Berz then argued on physical grounds that the dispersion relation should be independent of time. Consequently, only those velocity functions, $C(u_z)$, may be chosen which do not invalidate this assumption. Such a function is

$$(1.22) \quad C(u_z) = \frac{E_o C_o}{(\frac{w}{k} - u_z)} \; .$$

Integrating in the complex plane yields

$$(1.23)\quad \left(\frac{k}{w_p}\right)^2 = \int_{-\infty}^{\infty} \frac{\frac{dF_o}{du_z}\, du_z}{(u_z - \frac{w}{k})} \pm i\pi \frac{kC_o}{n_o}$$

where the sign of the second terms depends on the path of integration and the imaginary part of w/k is assumed to be zero. The value of C_o may be obtained by considering f(z,u,t) at t = 0. We have

$$(1.24)\quad f(z,u_z,t) = \frac{ieE_o e^{ikz}}{m(ku_z - w)} \left[n_o \left(\frac{dF_o}{du_z}\right) e^{-iwt} - kC_o e^{-iku_z t} \right]$$

At t = 0, this becomes

$$(1.25)\quad f(z,u_z,0) = \frac{ieE_o e^{ikz}}{m(ku_z - w)} \left[n_o \frac{dF_o}{du_z} - kC_o \right] .$$

Assuming that f(z, w/k, 0) $\neq$ 0, Berz arrived at the conclusion that

$$(1.26)\quad C_o = \frac{n_o}{k} \frac{dF_o(w/k)}{du_z}$$

which implies that

$$(1.27)\quad \left(\frac{k}{w_p}\right)^2 = \int_{-\infty}^{\infty} \frac{\frac{dF_o}{du_z}}{(u_z - \frac{w}{k})} du_z \pm i\pi \frac{dF_o(w/k)}{du_z} .$$

The work of Bohm and Gross would lead us to conclude that

$$(1.28) \quad \left(\frac{k}{w_p} \right)^2 = \int_{-\infty}^{\infty} \frac{\frac{dF_o}{du_z}}{(u_z - w/k)} \, du_z$$

where the initial conditions are neglected. There exist no physical reasons justifying the assumption that $\frac{dF_o(w/k)}{du}$ be zero. It is, however, a direct result of applying the substitution method to the problem.

Finally, we may criticize Berz on his choice of the function, $C(u_z)$. Any function eliminating time from the dispersion relation would be valid; another valid choice would be

$$(1.29) \quad C(u_z) = e^{-\alpha |(u)|} \frac{E_o C_o}{(w/k - u_z)}$$

or

$$(1.30) \quad C(u_z) = E_o C_o \, G(u_z) \, \delta(u_z - w/k)$$

where $G(u_z)$ is any function of u_z.

Indeed, the number of valid choices is limitless and each new choice gives rise to a new dispersion relation.

As we can see from the above, the Ansatz method and Berz's approach leave much to be desired in the solution of the transport equation.

1.7 Stationary Phase

Van Kampen [van Kampen, 1955] proposed that the difficulty in dealing with a pole at $w = ku$ has nothing to do with the physics of the problem but rather is due to the inability of the mathematical procedures used to treat the case of a continuous velocity distribution. His paper offered a mathematical technique that permits the use of the so called stationary wave approach in the case of a continuous non vanishing velocity distribution, $n_o F_o(u)$. Van Kampen linearized the transport equation in the usual fashion and obtained for the collisionless case

$$(1.31) \quad \frac{\partial f}{\partial t}(\bar{r},\bar{u},t) + \bar{u}\cdot\overline{\nabla}_r f + \frac{e}{m}\bar{E}\cdot\overline{\nabla}_u (n_o F_o) = 0$$

where

$$(1.32) \quad \bar{E}(\bar{r},t) = 4\pi e \int f(\bar{r},\bar{u},t)\, du^3.$$

Letting

$$(1.33) \quad f(\bar{r},\bar{u},t) = g(\bar{u})e^{i(\bar{k}\cdot\bar{r} - wt)}$$

and inserting into the linearized equation gives

$$(1.34) \quad (-w + \bar{k}\cdot\bar{u})g(\bar{u}) + \frac{e^2 n}{m}\circ\overline{\nabla}_u F_o\left(\frac{-4\pi\bar{k}}{k^2}\right)\int g(\bar{u})\, du^3 = 0.$$

Van Kampen further simplified his equations by assuming that:

(a) k is in the z direction

(b) the equilibrium velocity distribution is isotropic i.e.

(1.35) $F_o(\bar{u}) = F(|\bar{u}|)$

or

(1.36) $\overline{\nabla}_u F = \frac{\bar{u}}{u} \frac{\partial F_o}{\partial u}$.

This meant that

(1.37) $(-w+ku_z)g(\bar{u}) - \frac{4\pi n_o e^2}{m} \frac{\partial F_o}{\partial u} \int g(\bar{u})\, du^3 = 0.$

He then defined

(1.38) $\bar{g}(u_z) = \iint g(\bar{u})\, du_x\, du_y$

which implied that

(1.39) $(w-ku_z)\bar{g}(u_z) = \frac{8\pi^2 n_o e^2}{mk} u_z F_o(u_z)$

where the result that

(1.40) $\iint \frac{1}{u} \frac{\partial F_o}{\partial u} du_x\, du_y = -2\pi F_o(u_z)$

and definition that

(1.41) $\int \bar{g}(u_z)\, du_z = 1$

were used.

It is at this point that the problem requires some special attention. There had been several " a priori" prescriptions given for the integration across the pole but every different approach gave a different distribution function. Since every prescription consistent with the differential equation and the initial conditions is a valid solution to the transport equation and since the latter has been linearized, it follows that a superposition of all stationary phase solutions is also a solution. Van Kampen's contribution came as a result of a basic change in the consideration of solutions. Extending allowable solution to distributions in the sense of L. Schwartz [Schwartz, 1950], he allowed solutions of (1.39) to take the form

$$\bar{g}(u_z) = \frac{8\pi^2 e^2 n_o}{mk} u_z F_o(u_z) \left[P \frac{1}{w-ku_z} + \lambda\delta(w-ku_z) \right] \tag{1.42}$$

where P indicates that the Cauchy principal value of the resulting integral is to be taken and the symbol, $\delta(w-ku_z)$ represents the Dirac delta function. The algebraic form of the function $\lambda(w,k)$ is determined by the normalization condition expressed by equation (1.41).

Letting the distribution function assume such a character might at first sight seem somewhat unorthodox but its form can be readily justified. Consider the Boltzmann equation for the case where collisions are important and assume that the collision term is of the form

$$(1.43) \quad \left.\frac{\partial F}{\partial t}\right|_{coll} = \pm c\, f(\bar{r},\bar{u},t)$$

where c = constant.

Applying the very same procedure as outlined above would give

$$(1.44) \quad (w - ku_z \mp ic)\bar{g}(u_z) = \frac{8\pi^2 e^2 n_o}{km} u_z F_o(u_z).$$

In the limit of zero collisions, this becomes

$$(1.45) \quad \bar{g}(u_z) = \lim_{c \to 0} \left[\frac{\frac{8\pi^2 e^2 n_o}{mk} u_z F_o(u_z)}{(w - ku_z \mp ic)} \right] =$$

$$(1.46) \quad = \frac{8\pi^2 e^2 n_o}{mk} u_z F_o \left[P \frac{1}{w - ku_z} \pm i\pi\delta(w - ku_z) \right]$$

[Dirac, 1958] . This is at least a plausibility argument justifying the form of the distribution function, $\bar{g}(u_z)$.

Returning to van Kampen's stationary phase solution (1.42) and applying the normalization condition (1.41), we obtain

$$(1.47) \quad 1 = \frac{8\pi^2 e^2 n_o}{mk} P \int \frac{u_z F_u(u_z)}{(w-ku_z)} du_z + \lambda(w,k)$$

which gives a prescription for determining $\lambda(w,k)$. The appearance of the new parameter, $\lambda(w,k)$, makes it possible to satisfy the consistency equation

$$(1.41) \quad \int_{-\infty}^{\infty} \bar{g}(u_z)\, du_z = 1$$

without having to relate w to k.

Finally, van Kampen was able to show that the perturbation distribution was given by a superposition of stationary waves such that

$$(1.48)\quad \bar{f}(z,u_z,t) = \iint_{-\infty}^{\infty} C(k,v)\bar{g}_{k,v}(u_z)e^{ik(z-vt)}dk\,dv$$

where the only spatial variation is in the z direction and C(k,v) is an expansion coefficient the existence of which can be shown and whose algebraic form depends on the initial conditions.

Although van Kampen's work represents a significant initial step in the stationary wave approach to the transport equation, it is limited by a rather unphysical assumption, namely, that $F_o(\frac{w}{k})$ never be zero and contradicts the conclusion made by Landau on the existence of a dispersion relation. The clarification of these points together with a major revamping of the method of stationary phase was the subject of a paper presented by K. M. Case [Case, 1959].

In the remaining part of this work, we will make an effort to bring to light the major points of Case's method and to show how a modification of the work done by Shure [Shure, 1962], an enlargement of Case's original paper, can lead to an extension of the method's range of applicability.

II THE SOLUTION OF THE INITIAL VALUE PROBLEM FOR THE CASE OF A CONSTANT OUTSIDE IMPRESSED MAGNETIC FIELD

The first complete application of the normal mode analysis to the treatment of the transport equation is due to Case [Case, 1959] and was later amplified by Shure [Shure, 1962] . It is the purpose of this chapter to:

a.) Delineate the major features of the normal mode technique and Shure's contribution.

b.) Extend the work of Shure to the treatment of an initial value problem for the case of a plasma immersed in a constant outside imposed magnetic field whose direction is that of the longitudinal plasma oscillations. Furthermore, we must stipulate that the medium is characterized by an over-all charge neutrality but may possess local inhomogeneities in the electron concentration.

c.) Indicate the relationship among Shure's work, the present endeavor, and the more limited problem that Case considered.

We can best achieve the above stated purposes by first outlining the initial portion of Shure's work.

2.1 Linearization of the Transport Equation

Restricting our attention to those physical systems that may be treated by the classical statistical mechanics, we may find a first order approximation to many of the system's properties by solving the transport equation together with the four Maxwell relations. The singlet density transport equation derived by Vlasov [Vlasov, 1945] is

$$(2.1) \qquad \frac{\partial F(\bar{r},\bar{u},t)}{\partial t} + \bar{u}\cdot\overline{\nabla}_r F + \bar{a}\cdot\overline{\nabla}_u F = - \left.\frac{\delta F}{\delta t}\right|_{coll}$$

where the meaning of each term was given in equation (1.3). In order to gain a clearer insight into the inherent limitations of our method, we shall elaborate somewhat on the physical conditions that are required. The acceleration, $\bar{a}(\bar{r},t)$, is said to be smooth and the potential, $\varphi(\bar{r},t)$, smeared out. This imposes some rather strong restrictions on the type of system that we may consider. We must stipulate that $n\lambda_D^3 \gg 1$, i.e. the product of the undisturbed electron density and the "volume of effective interactions" defined by the Debye length cubed must be much larger than unity. Each charged particle therefore sees many other charged particles in its surrounding volume of magnitude, λ_D^3.

In what follows, $\left.\frac{\delta F}{\delta t}\right|_{coll.}$ will not appear. We may interpret this as signifying that binary collisions are extremely infrequent or that we are not interested in variations of $F(\bar{r},\bar{u},t)$ that occur in the time interval

$$(2.2) \quad (\delta t)_{coll.} = \frac{\ell}{|\bar{v} - \bar{v}'|}$$

Here, ℓ is a characteristic length of the system satisfying the condition that $\ell \ll \lambda_D$ and $\bar{v}$ and $\bar{v}'$ are the electron velocities before and after the violent interaction of a binary collision.

In passing, we may add that a collision term of the type

$$(2.3) \quad \left.\frac{\delta F}{\delta t}\right|_{coll.} = -cf(\bar{r},\bar{u},t)$$

where c is a constant and $f(\bar{r},\bar{u},t)$ is a distribution function representing a small disturbance from the equilibrium state, may also be considered although we shall not do so here.

Finally, we arrive at the strongest restriction. The equation (2.1) is non-linear since the acceleration, $\bar{a}$, varies as the electric field, which in turn varies as the distribution function, $F(\bar{r},\bar{u},t)$. In order to avoid the complications of non-linear techniques and still consider many meaningful physical problems, we shall obtain a suitably linearized transport equation by placing further restrictions on the nature of the ionized system. Assuming the function, $F(\bar{r},\bar{u},t)$, to represent an electron distribution that is not far removed from the equilibrium distribution, $n_o F_o(\bar{u})$, and describing those relatively few electrons whose distribution differs from those in the equilibrium state by $f(\bar{r},\bar{u},t)$, we write

$$F(\bar{r},\bar{u},t) = n_o F_o(\bar{u}) + f(\bar{r},\bar{u},t) \tag{2.4}$$

where

$$n_o F_o(\bar{u}) \gg f(\bar{r},\bar{u},t) \tag{2.5}$$

and n_o is the density of electrons in the equilibrium state. This supposes that the number of electrons giving rise to the local electric and magnetic fields is very small compared to the total number of electrons in the system. Inserting these conditions into the transport equation and remembering that $\left.\frac{\delta F}{\delta t}\right|_{coll.}$ has been set equal to zero gives

$$\frac{\partial f(\bar{r},\bar{u},t)}{\partial t} + \bar{u}\cdot\overline{\nabla}_r f + \bar{a}\cdot\overline{\nabla}_u(n_o F_o + f) = 0 \tag{2.6}$$

where the condition that $n_o F_o \gg f$ has not as yet been used. Summarily dropping f with respect to $n_o F_o$ for the case of a local acceleration given by

$$\bar{a} = \frac{e}{m}\left[\bar{E}(\bar{r},t) + \frac{\bar{u}}{c}\,(B_o\hat{k} + \bar{B}(\bar{r},t))\right] \tag{2.7}$$

would give incorrect results. In the above:

e = the electric charge and therefore represents a negative quantity.

m = the mass of the electron.

$\bar{E}(\bar{r},t)$ = the electric field caused by the local inhomogneities in the electronic concentration.

$\bar{B}(\bar{r},t)$ = the associated or self-consistent magnetic field due to the electron currents and related to $\bar{E}(\bar{r},t)$ via the Maxwell equations.

$B_o\hat{k}$ = a constant outside imposed magnetic field whose magnitude is such that it remains constant throughout the ionized medium and whose direction is that of the longitudinal plasma oscillations i.e. the k or z direction.

It is important to note at this point that Case [Case, 1959] drops the self-consistent magnetic field in his treatment. Although, this is usually quite small due to the u/c factor, the present approach will include the effect.

Imposing two more limitations on the system permits an adequate treatment of the acceleration term in the transport equation. Supposing that the initial distribution $F_o(\bar{u})$ is isotropic in velocity and that the perturbed distribution, $f(\bar{r},\bar{u},t)$, varies spatially only in the argument z, we write

(2.8) $$F_o(\bar{u}) = F_o(|\bar{u}|)$$

(2.9) $$f(\bar{r},\bar{u},t) = f(z,\bar{u},t)$$

where equation (2.9) implies that all field quantities vary only in z and t, i.e.

$$(2.10)\quad \bar{E} = \bar{E}(z,t)$$

$$(2.11)\quad \bar{B} = \bar{B}(z,t).$$

Keeping the above restrictions in mind, we may write the full $\bar{a}\cdot\overline{\nabla}_u(n_oF_o + f)$ term as

$$(2.12)\quad \bar{a}\cdot\overline{\nabla}_u(n_oF_o+f) = \frac{n_oe}{m}\,\bar{E}\cdot\overline{\nabla}_uF_o + \frac{n_oe}{m}\left[\frac{\bar{u}}{c}\times(B_o\hat{k}+\bar{B})\right]\cdot\overline{\nabla}_uF_o$$

$$+\frac{e}{m}\bar{E}\cdot\overline{\nabla}_uf + \frac{e}{mc}(\bar{u}\times B_o\hat{k})\cdot\overline{\nabla}_uf + \frac{e}{mc}\,(\bar{u}\times\bar{B})\cdot\overline{\nabla}_uf \quad .$$

The third and fifth terms on the right hand side of equation (2.12) are dropped since they involve the product of two first order terms. The second term on the right hand side is identically zero due to the isotropic nature of F_o, i.e.

$$(2.13)\quad \left[\bar{u}\times(B_o\hat{k}+\bar{B})\right]\cdot\overline{\nabla}_uF_o = -(B_o\hat{k}+\bar{B})\cdot(\bar{u}\times\overline{\nabla}_uF_o).$$

But, for an istropic F_o,

$$(2.14)\quad \bar{u}\times\overline{\nabla}_uF_o = 0.$$

We may therefore write the linearized Boltzmann equation for the case of a collisionless plasma under the influence of a constant magnetic field as

$$(2.15)\quad \frac{\partial f(z,\bar{u},t)}{\partial t} + u_z\frac{\partial f}{\partial z} + \frac{eB_o}{mc}(\bar{u}\times\hat{k})\cdot\overline{\nabla}_uf = -\frac{n_oe}{m}\bar{E}(z,t)\cdot\overline{\nabla}_uF_o.$$

It is interesting to note that a constant outside applied electric field, $\bar{E}_o$, would have introduced an inhomogeneous term in the Boltzmann equation, i.e. $\bar{E}_o \cdot \bar{\nabla}_u F_o$. The assumed isotropic nature of the equilibrium distribution is the characteristic that permits such an introduction in the magnetic field case.

Finally, we must add that our linearization of the transport equation is valid only for certain values of B_o. In equation (2.12), we dropped the second order terms with respect to $\frac{e}{mc}(\bar{u} \times B_o \hat{k}) \cdot \bar{\nabla}_u f$; this implies that

$$(2.16) \quad \frac{\bar{u}}{c} \times B_o \hat{k} \gg \bar{E}(z,t) + \frac{\bar{u}}{c} \times \bar{B}(z,t).$$

Clearly, our equations cannot hold for the case where B_o is so small that the left hand side of (2.16) becomes of the same order of magnitude as the right hand side. They do hold, however, when $B_o = 0$ for then the three last terms in equation (2.12) vanish.

2.2 The Maxwell Equations in Component Form

The perturbed distribution function, $f(z,\bar{u},t)$ gives rise to the two field quantities $\bar{E}(z,t)$ and $\bar{B}(z,t)$ and is related to them via the four Maxwell equations. In component form, these are

$$(2.17) \quad \frac{\partial E_z}{\partial z} = 4\pi e \int_{-\infty}^{\infty} f(z,\bar{u},t)\, du^3$$

$$(2.18) \quad \frac{\partial B_z}{\partial z} = 0$$

$$(2.19) \quad \frac{\partial E_y}{\partial z} = \frac{1}{c} \frac{\partial B_x}{\partial t}$$

$$(2.20) \quad \frac{\partial E_x}{\partial z} = -\frac{1}{c} \frac{\partial B_y}{\partial t}$$

$$(2.21) \quad 0 = \frac{\partial B_z}{\partial t}$$

$$(2.22) \quad -\frac{\partial B_y}{\partial z} = \frac{1}{c} \frac{\partial E_x}{\partial t} + \frac{4\pi e}{c} \int_{-\infty}^{\infty} u_x f(z,\bar{u},t)\, du^3$$

$$(2.23) \quad \frac{\partial B_x}{\partial z} = \frac{1}{c} \frac{\partial E_y}{\partial t} + \frac{4\pi e}{c} \int_{-\infty}^{\infty} u_y f(z,\bar{u},t)\, du^3$$

$$(2.24) \quad 0 = \frac{1}{c} \frac{\partial E_z}{\partial t} + \frac{4\pi e}{c} \int_{-\infty}^{\infty} u_z f(z,\bar{u},t)\, du^3.$$

Combining equations (2.18) and (2.21) indicates that B_z = constant. Choosing any constant other than zero would, however, by physically meaningless since there exists in the plasma no process that gives rise to a constant magnetic field in the z direction.

2.3 Longitudinal and Transverse Mode Analysis

The above set of five coupled partial differential equations can be solved if instead of seeking solutions of the linearized transport equation as it stands, we attempt to solve a modified first moment of the latter. Thus, a solution is possible if we seek

$$(2.25)\quad g_x(z,u_z,t) \equiv \iint_{-\infty}^{\infty} u_x f(z,\bar{u},t)\, du_x\, du_y$$

$$(2.26)\quad g_y(z,u_z,t) \equiv \iint_{-\infty}^{\infty} u_y f(z,\bar{u},t)\, du_x\, du_y$$

$$(2.27)\quad g_z(z,u_z,t) \equiv \iint_{-\infty}^{\infty} u_z f(z,\bar{u},t)\, du_x\, du_y.$$

In the above, $g_x(z,u_z,t)$ and $g_y(z,u_z,t)$ are referred to as the transverse or x and y modes whereas $g_z(x,u_z,t)$ is called the longitudinal or z-mode. The differential equations satisfied by these functions can be obtained by multiplying the linearized transport equation by u_α where $\alpha = x,y,z$, and integrating the result over the range of u_x and u_y i.e. $-\infty \leq u_x,\ u_y \leq +\infty$. Performing these operations on equation (2.15) and integrating by parts yields

$$(2.28)\quad \frac{\partial g_x(z,u_z,t)}{\partial t} + u_z \frac{\partial g_x}{\partial z} + \Omega g_y = \frac{ne}{m} E_x(z,t)F(u_z)$$

$$(2.29)\quad \frac{\partial g_y(z,u_z,t)}{\partial t} + u_z \frac{\partial g_y}{\partial z} - \Omega g_x = \frac{ne}{m} E_y(z,t)F(u_z)$$

$$(2.30)\quad \frac{\partial g_z(z,u_z,t)}{\partial t} + u_z \frac{\partial g_z}{\partial z} = \frac{-ne}{m} u_z E_z(z,t)\frac{dF(u_z)}{du_z}$$

where Ω is $-\frac{eB_o}{mc}$, the cyclotron frequency

$$(2.31) \quad F(u_z) \equiv \iint_{-\infty}^{\infty} F_o(|\bar{u}|)\, du_x\, du_y$$

and where the fact that F_o is an even function of velocity with the characteristic that

$$(2.32) \quad F_o \Big|_{-\infty}^{\infty} = u_x F_o \Big|_{-\infty}^{\infty} = u_y F_o \Big|_{-\infty}^{\infty} = 0$$

was used. Similarly, we assume that

$$(2.33) \quad f \Big|_{-\infty}^{\infty} = u_x f \Big|_{-\infty}^{\infty} = u_y f \Big|_{-\infty}^{\infty} = 0$$

A comparison of the two transverse modes indicates that the intermode coupling factor depends directly on the presence of the constant, outside impressed magnetic field, B_o. The symmetry between the equations is expected since the only preferred direction in the system is the z or longitudinal oscillation direction. We also note that the z-mode of oscillation is totally unaffected by the presence of the magnetic field, $B_o\hat{k}$. The latter is expected, however, since there can exist no coupling between the " current", $g_z(z,u_z,t)$, and the field, B_o, i.e.

$$(2.34) \quad (u_z\hat{k}) \times (B_o\hat{k}) = 0$$

Let us discuss, at this point, the physical significance of the functions $g_x(z,u_z,t)$, $g_y(z,u_z,t)$ and $g_z(z,u_z,t)$. This can be done by considering the definition of electric current density i.e.

$$(2.35) \quad \bar{j}(z,t) \equiv e \int_{-\infty}^{\infty} \bar{u}\, f(z,\bar{u},t)\, du^3$$

$$(2.36) \quad \equiv e\, \bar{J}(z,t)$$

where $\bar{J}(z,t)$ represents the electron current density. However, we may consider $\bar{J}(z,t)$ to be the u_z integral of

$$(2.37) \quad \frac{\partial \bar{J}}{\partial u_z} \equiv \iint_{-\infty}^{\infty} \bar{u}\, f(z,\bar{u},t)\, du_x\, du_y$$

Consequently, we may interpret the functions $g_\alpha(z,u_z,t)$ defined previously as follows

$$(2.38) \quad g_x(z,u_z,t) = \hat{i}\cdot\frac{\partial \bar{J}}{\partial u_z}$$

$$(2.39) \quad g_y(z,u_z,t) = \hat{j}\cdot\frac{\partial \bar{J}}{\partial u_z}$$

$$(2.40) \quad g_z(z,u_z,t) = \hat{k}\cdot\frac{\partial \bar{J}}{\partial u_z}$$

This implies that $g_\alpha(z,u_z,t)$ is that component of electron current flowing along the α direction and due to all electrons whose u_z component of velocity lies in the range u_z to $u_z + du_z$.

2.4 Summary of the Transverse and Longitudinal Modes with Corresponding Component Maxwell Equations

The problem has, therefore, been reduced to solving the following sets of coupled equations

x-mode (transverse mode)

$$(2.41)\quad \frac{\partial g_x}{\partial t} + u_z \frac{\partial g_x}{\partial z} + \Omega g_y = \frac{ne}{m} E_x(z,t)\ F(u_z)$$

$$(2.42)\quad \frac{\partial E_x}{\partial z} = -\frac{1}{c}\frac{\partial B_y}{\partial t}$$

$$(2.43)\quad \frac{\partial B_y}{\partial z} = -\frac{1}{c}\frac{\partial E_x}{\partial t} - \frac{4\pi e}{c}\int_{-\infty}^{\infty} g_x\, du_z$$

y-mode (transverse mode)

$$(2.44)\quad \frac{\partial g_y}{\partial t} + u_z \frac{\partial g_y}{\partial z} - \Omega g_x = \frac{ne}{m} E_y(z,t)\ F(u_z)$$

$$(2.45)\quad \frac{\partial E_y}{\partial z} = \frac{1}{c}\frac{\partial B_x}{\partial t}$$

$$(2.46)\quad \frac{\partial B_x}{\partial z} = \frac{1}{c}\frac{\partial E_y}{\partial t} + \frac{4\pi e}{c}\int_{-\infty}^{\infty} g_y\, du_z$$

z-mode (longitudinal mode)

$$(2.47)\quad \frac{\partial g_z}{\partial t} + u_z \frac{\partial g_z}{\partial z} = \frac{-ne}{m} u_z\, E_z(z,t)\, \frac{dF(u_z)}{du_z}$$

$$(2.48)\quad \frac{\partial E_z}{\partial t} = -4\pi e \int_{-\infty}^{\infty} g_z\, du_z$$

$$(2.49) \quad \frac{\partial E_z}{\partial z} = 4\pi e \int_{-\infty}^{\infty} \frac{g_z}{u_z} \, du_z$$

$$(2.50) \quad B_z = 0$$

2.5 The Normal Mode Approach to the Solution of the Initial Value Problem for the Longitudinal Mode

The initial value problem for the longitudinal mode has been solved by Case. It is briefly outlined here in terms of our notation for it brings to light the main concepts of the normal mode approach. Consider a spatial Fourier decomposition of the equations (2.47) and (2.49) such that

$$(2.51) \quad g_z(z,u_z,t) \sim u_z \, \bar{g}_k(u_z,t) \, e^{ikz}$$

$$(2.52) \quad E_z(z,t) \sim E_k(t) \, e^{ikz}$$

Equations (2.47) and (2.49) then become

$$(2.53) \quad \frac{\partial \bar{g}_k(u_z,t)}{\partial t} + iku_z\bar{g}_k = \frac{-ne}{m} E_k(t) \, F'(u_z)$$

$$(2.54) \quad ikE_k(t) = 4\pi e \int_{-\infty}^{\infty} \bar{g}_k(u_z,t) \, du_z$$

where

$$(2.55) \quad F'(u_z) \equiv \frac{dF(u_z)}{du_z}$$

Making the assumption that the functions have a time dependence of the type $e^{-ik\nu t}$ i.e.

$$(2.56) \quad \bar{g}_k(u_z,t) \sim g_{k,\nu}(u_z)e^{-ik\nu t}$$

$$(2.57) \quad E_k(t) \sim E_{k,\nu}\, e^{-ik\nu t}$$

and introducing this into (2.53) and (2.54) yields

$$(2.58) \quad (u_z - \nu)g_\nu(u_z) = \frac{w_p^2}{k^2} F'(u_z) \int_{-\infty}^{\infty} g_\nu(u_z)\, du_z$$

$$(2.59) \quad E_\nu = \frac{4\pi e}{ik} \int_{-\infty}^{\infty} g_\nu(u_z)\, du_z$$

where the k subscript is dropped for convenience. There exists some degree of arbitrariness at this point concerning the value to give the integral appearing in equations (2.58) and (2.59). Since equation (2.58) is linear in $g_\nu(u_z)$, we may normalize the integral to any function of k and ν, say

$$(2.60) \quad \int_{-\infty}^{\infty} g_\nu(u_z)\, du_z = \gamma(k,\nu)$$

For convenience, however, we let $\gamma(k,\nu) = 1$. This means that we are left with

$$(2.61) \quad (u_z - \nu)g_\nu(u_z) = \frac{w_p^2}{k^2} F'(u_z)$$

$$(2.62) \quad E_\nu = \frac{4\pi e}{ik}$$

We must stop at this point to consider the ultimate goal of this procedure. We seek a set of functions $\left\{g_\nu (u_z)\right\}$ which is complete in u_z and whose component functions, $g_\nu (u_z)$, are the so-called normal mode or fixed frequency solutions of (2.61). The set of normal mode solutions is complete in the sense that any function, $G(u_z)$, may be expressed as a sum of discrete normal mode solutions plus an integral over the continuous normal mode solutions, i.e.

$$(2.63) \quad G(u_z) = \sum_\alpha a_\alpha(k, \nu_\alpha) \cdot g_{\nu_\alpha}(u_z) + \int_{-\infty}^{\infty} A_k(\nu) g_\nu (u_z) du_z$$

In order to find the expansion coefficients, $a_\alpha(k, \nu_\alpha)$ and $A_k(\nu)$, however, we must also find the normal mode solutions to the so-called adjoint which is chosen such that its solutions, $g^+_{\nu'}(u_z)$, are orthogonal to the solutions, $g_\nu (u)$, over the range $-\infty \leq u_z \leq \infty$ whenever the normal mode indices, ν and ν' are different. Returning now to equation (2.61), we see that its solution may be segregated into various classes.

Class 1.

Here, ν is complex with a non-zero imaginary part. Therefore, we have

$$(2.64) \quad g_{\nu_j}(u_z) = \frac{w_p^2}{k^2} \frac{F'(u_z)}{(u_z - \nu_j)}$$

where ν_j is a root of

$$(2.65) \quad 1 = \frac{w_p^2}{k^2} \int_{-\infty}^{\infty} \frac{F'(u_z)\, du_z}{(u_z - \nu_j)}$$

Class 2a.

Here ν is real. Letting the normal mode solution assume the character of a distribution in the sense of Schwartz, [Schwartz, 1950] we obtain

$$(2.66) \quad g_\nu(u_z) = \frac{w_p^2}{k^2} P\left\{\frac{F'(u_z)}{u_z - \nu}\right\} + \lambda(\nu)\delta(u_z - \nu)$$

where P indicates that the Cauchy principal value of the resulting integral must be taken and $\delta(u_z - \nu)$ is the Dirac delta function. The function $\lambda(\nu)$ is determined by

$$(2.67) \quad 1 - \frac{w_p^2}{k^2} P \int_{-\infty}^{\infty} \frac{F'(u_z)}{(u_z - \nu)}\, du_z = \lambda(\nu)$$

Class 2b.

Here, ν is real, but $F'(\nu) = 0$. We again obtain the solution indicated for class 2a but, in this case, the principal value sign is not needed.

Class 2c.

Again, ν is real but $F'(\nu) = \lambda(\nu) = 0$. Consequently the solution is

$$(2.68) \quad g_{\nu_i}(u_z) = \frac{w_p^2}{k^2} \frac{F'(u_z)}{(u_z - \nu_i)}$$

and

$$(2.69) \quad 1 = \frac{w_p^2}{k^2} \int_{-\infty}^{\infty} \frac{F'(u_z)}{(u_z - \nu_1)} \, du_z$$

These solutions are analogous to the discrete solutions for the case of a complex value for ν.

2.5.1 The Adjoint Equation

The associated adjoint function satisfies the following algebraic equation

$$(2.70) \quad (u_z - \nu')g^+_{\nu'}(u_z) = \frac{w_p^2}{k^2} \int_{-\infty}^{\infty} F'(u_z)g^+_{\nu'}(u_z) \, du_z$$

where we normalize the integral to unity. Consequently, we have

$$(2.71) \quad (u_z - \nu')g^+_{\nu'}(u_z) = \frac{w_p^2}{k^2}$$

2.5.2 Orthogonality

The above adjoint equation was chosen such that its solution $g^+_{\nu'}(u_z)$ would be orthogonal to $g_\nu(u_z)$ over the range of integration, $-\infty \le u_z \le +\infty$. This may readily be seen as follows. Multiply (2.61) by $g^+_{\nu'}(u_z)$ and (2.71) by $g_\nu(u_z)$ and subtract the results; this gives

$$(2.72) \quad (\nu' - \nu)g_\nu(u_z)g^+_{\nu'}(u_z) = \frac{w_p^2}{k^2}\left[F'(u_z)g^+_{\nu'}(u_z) - g_\nu(u_z)\right]$$

Integrating over the range of velocity of u_z yields

$$(2.73) \qquad (\nu' - \nu) \int_{-\infty}^{\infty} g_{\nu}(u_z) g^{+}_{\nu'}(u_z)\, du_z = 0.$$

Consequently, we may write

$$(2.74) \qquad \int_{-\infty}^{\infty} g_{\nu}(u_z)\, g^{+}_{\nu'}(u_z)\, du_z = 0 \quad ; \qquad \nu \neq \nu'$$

This demonstrates orthogonality.

2.5.3 Solutions of the Adjoint Equation

The equation (2.71) like equation (2.61) has several solutions that must be separated into various classes.

Class 1

Here, ν is complex with a non-zero imaginary part. Therefore, we have

$$(2.75) \qquad g^{+}_{\nu_j}(u_z) = \frac{w_p^2}{k^2} \frac{1}{(u_z - \nu_j)}$$

where ν_j is a discrete root of

$$(2.76) \qquad 1 = \int_{-\infty}^{\infty} F'(u_z) g^{+}_{\nu_j}(u_z) du_z = \frac{w_p^2}{k^2} \int_{-\infty}^{\infty} \frac{F'(u_z) du_z}{(u_z - \nu_j)}$$

This is seen to be in agreement with the root as found in equation (2.65).

Class 2a

Here, ν is real. Again, admitting distributions in the sense of Schwartz, we have

$$(2.77)\quad g^+_\nu(u_z) = \frac{w_p^2}{k^2} P\left\{\frac{1}{u_z-\nu}\right\} + \lambda^+(\nu)\delta(u_z-\nu)$$

where $\lambda^+(\nu)$ is determined by

$$(2.78)\quad 1 = \frac{w_p^2}{k^2} P\int_{-\infty}^{\infty} \frac{F'(u_z)\, du_z}{(u_z-\nu)} + F'(\nu)\,\lambda^+(\nu)$$

Consequently,

$$(2.79)\quad \lambda^+(\nu) = \frac{\lambda(\nu)}{F'(\nu)}$$

Class 2b

Here, ν is again real but $F'(\nu) = 0$. We must also stipulate that $\lambda(\nu) \neq 0$. The solution in this case cannot be similar in form to that of class 2a since the condition expressed by equation (2.79) cannot be satisfied. A consistent solution is

$$(2.80)\quad g^+_\nu(u_z) = \delta(u_z-\nu)$$

Inserting this into equation (2.70) gives

$$(2.81)\quad (u_z-\nu)\delta(u_z-\nu) = \frac{w_p^2}{k^2}\int_{-\infty}^{\infty} F'(u_z)\delta(u_z-\nu)\, du_z$$

$$= F'(\nu) = 0$$

This is an identity. Therefore, our delta distribution solution is consistent with the adjoint equation.

Class 2c

Again, ν is real but $F'(\nu) = \lambda(\nu) = 0$. Consequently, the solution is

$$(2.82) \quad g^{+}_{\nu_i}(u_z) = \frac{w_p^2}{k^2} \frac{1}{(u_z - \nu_i)}$$

where ν_i is a discrete root of

$$(2.83) \quad 1 = \frac{w_p^2}{k^2} \int_{-\infty}^{\infty} \frac{F'(u_z)\, du_z}{(u_z - \nu_i)}$$

2.5.4 The Normalization Coefficients

Returning to the results of the orthogonality relation, equation (2.74), we now discuss evaluation when $\nu = \nu'$ for the discrete and continuous cases.

Discrete Case:

For the case where $\nu = \nu'$, equation (2.74) becomes

$$(2.84) \quad \int_{-\infty}^{\infty} g_{\nu_\alpha}(u_z)\, g^{+}_{\nu_\alpha}(u_z)\, du_z = N(\nu_\alpha)$$

where ν_α represents the discrete normal mode index and may be real or complex and $N(\nu_\alpha)$ is the normalization coefficient for the discrete case. Inserting the proper normal mode solutions into the above gives

$$(2.85) \quad N(\nu_\alpha) = \left(\frac{w_p^2}{k^2} \right)^2 \int_{-\infty}^{\infty} \frac{F'(u_z)}{(u_z - \nu_\alpha)^2}\, du_z$$

where ν_α represents the real or complex discrete roots.

Continuous Case:

The normalization coefficient for the continuous case may be derived in the same manner as that for the discrete case but a little care is needed in performing the integral over the product of the two singular functions, $g_{\nu}(u_z)$ and $g^{+}_{\nu}(u_z)$. Symbolically, we find

$$(2.86) \qquad \int_{-\infty}^{\infty} g_{\nu}(u_z) g^{+}_{\nu'}(u_z)\, du_z = N(\nu)\delta(\nu - \nu')$$

where

$$(2.87) \qquad N(\nu) = \frac{\lambda^2(\nu) + \left[\pi \frac{w_p^2}{k^2} F'(\nu)\right]^2}{F'(\nu)}$$

Extreme care must be exercised in the interpretation of equation (2.86). A rather detailed treatment of the normalization coefficient for the continuous case illustrating the use of the Poincaré-Bertrand transformation [Muskhelishvili, 1953] will be given in the consideration of the x and y mode solutions.

2.5.5 The Expansion Coefficients

Case [Case, 1959] was able to show that the set of functions, $\{g_{\nu}(u_z)\}$ is complete in u_z such that any function of u_z may be expressed in terms of these. Specifically, he showed the existence of the expansion coefficiencts, $a_\alpha(k, \nu_\alpha)$ and $A_k(\nu)$, in the expression

$$(2.88) \qquad \bar{g}_k(u_z, 0) = \sum_{\alpha} a_\alpha g_{k,\nu_\alpha}(u_z) + \int_{-\infty}^{\infty} A_k(\nu') g_{k,\nu'}(u_z)\, d\nu'$$

where $\bar{g}_k(u_z,0)$ is the initial value of $\bar{g}_k(u_z,t)$. The expansion coefficients may therefore be obtained in a straightforward manner.

Determination of $a_\alpha(k, \nu_\alpha)$:

Multiplying (2.88) by $g^+_{\nu_\alpha}(u_z)$ and integrating the results over u_z gives

$$(2.89) \quad a_\alpha(k, \nu_\alpha) = \frac{1}{N(\nu_\alpha)}\int_{-\infty}^{\infty} g^+_{\nu_\alpha}(u_z)\bar{g}_k(u_z,0)\, du_z$$

Determination of $A_k(\nu)$:

Similarly, we multiply (2.88) by $g^+_\nu(u_z)$ and integrate over u_z; this yields

$$(2.90) \quad A_k(\nu) = \frac{1}{N(\nu)}\int_{-\infty}^{\infty} g^+_\nu(u_z)\, \bar{g}_k(u_z,0)\, du_z$$

2.5.6 Final Solution to the Longitudinal Oscillation Problem

We are now in a position to express the answer to the longitudinal oscillation problem. At any time, $t \geq 0$, we may write

$$(2.91) \quad \bar{g}_k(u_z,t) = \sum_\alpha a_\alpha\, g_{\nu_\alpha,k}\, e^{-ik\nu_\alpha t} + \int_{-\infty}^{\infty} A_k(\nu)\, g_{\nu,k}(u_z)\, e^{-ik\nu t}\, d\nu$$

and remembering that

$$(2.92)\quad \varphi_k(t) = \frac{4\pi e}{k^2} \int_{-\infty}^{\infty} \bar{g}_k(u_z,t)\, du_z$$

and

$$(2.93)\quad 1 = \int_{-\infty}^{\infty} g_\nu(u_z)\, du_z$$

we obtain

$$(2.94)\quad \varphi_k(t) = \frac{4\pi e}{k^2}\left[\sum_\alpha a_\alpha e^{-ik\,\nu_\alpha t} + \int_{-\infty}^{\infty} A_k(\nu)e^{-ik\nu\, t} d\nu\right]$$

Taking the inverse Fourier space transform of the above gives us therefore

$$(2.95)\quad g'(z,u_z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{g}_k(u_z,t)\, e^{-ikz}\, dk$$

$$(2.96)\quad \varphi(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_k(t)\, e^{-ikz}\, dk$$

Actually, we must multiply (2.95) by u_z to obtain $g_z(z,u_z,t)$ i.e.

$$(2.97)\quad g_z(z,u_z,t) = u_z\, g'(z,u_z,t)$$

This then completes the solution for we now have the " current", $g_z(z,u_z,t)$, and the potential, $\varphi(z,t)$, as functions of their arguments. The last quantity also implies that we know $E(z,t)$.

This brief outline on the normal mode analysis of the longitudinal oscillation problem illustrates the major features of the method and should render our treatment of the transverse modes somewhat easier.

2.6 Right and Left Circularly Polarized Modes

The coupled nature of the transverse modes of oscillation requires that we attempt a simultaneous solution. This can be accomplished by letting

$$(2.98)\quad E_{\pm}(z,t) = E_x(z,t) \pm iE_y(z,t)$$

$$(2.99)\quad B_{\pm}(z,t) = B_x(z,t) \pm iB_y(z,t)$$

$$(2.100)\quad g_{\pm}(z,u_z,t) = g_x(z,u_z,t) \pm ig_y(z,u_z,t)$$

where the plus mode is referred to as " right circularly polarized" and the minus mode is called " left circularly polarized". Using these definitions and combining the x and y modes gives

$$(2.101)\quad \frac{\partial g_{\pm}}{\partial t} + u_z \frac{\partial g_{\pm}}{\partial z} \mp \Omega g_{\pm} = \frac{ne}{m} E_{\pm} F(u_z)$$

$$(2.102)\quad \frac{\partial E_{\pm}}{\partial z} = \pm \frac{i}{c} \frac{\partial B_{\pm}}{\partial t}$$

$$(2.103)\quad \pm i \frac{\partial B_{\pm}}{\partial z} = \frac{1}{c} \frac{\partial E_{\pm}}{\partial t} + \frac{4\pi e}{c} \int_{-\infty}^{\infty} g_{\pm}\, du_z$$

These three equations are now in a form that permits the use of the normal mode technique.

The above development leading to the formulation of an oscillatory mode and two circularly polarized modes is simply an amplification of the work done by Shure [Shure, 1962]. The importance of this background formulation coupled to the rather limited accessibility of Shure's dissertation makes this repetition desirable. It is at this point, however, that the development of Shure's work and that of the present thesis diverge.

2.7 The Initial Value Problem for the Transverse Modes and the Matrix Formulation

The purpose of Shure's dissertation consisted in showing that the functions $g_{\pm}(z,u,t)$ and the field quantities, $E_{\pm}(z,t)$, $B_{\pm}(z,t)$ could be determined provided that their values were known at certain boundaries for all times. In solving the above boundary value problem, he Fourier analyzed the time variable and assumed that the space variable of the quantities $g_{\pm}$, $E_{\pm}$ and $B_{\pm}$ varied as $e^{i(w/\nu)z}$, where ν is the normal mode (van Kampen calls it stationary mode) index. Combining the three quantities, $g_{\pm,\nu}$, $E_{\pm,\nu}$, $B_{\pm,\nu}$, into a so-called "state vector", $\Psi_{\pm,\nu}$, and defining a new " adjoint state vector" ,

$\Psi^{+}_{\pm,\nu}$, where

$$(2.104)\qquad \Psi_{\pm\nu} \equiv \begin{bmatrix} g_{\pm\nu} \\ E_{\pm\nu} \\ B_{\pm\nu} \end{bmatrix} \quad ; \quad \Psi^{+}_{\pm\nu} \equiv \begin{bmatrix} g^{+}_{\pm\nu} \\ E^{+}_{\pm\nu} \\ B^{+}_{\pm\nu} \end{bmatrix}$$

such that the two vectors would be orthogonal whenever their normal mode indices differed, he was able to show the existence of a complete set of functions. This was the object of his work for he could then express

$$(2.105)\qquad \Psi_{\pm} \equiv \begin{bmatrix} g_{\pm} \\ E_{\pm} \\ B_{\pm} \end{bmatrix} \qquad \text{(no normal mode index)}$$

in terms of the complete set of functions $\left\{ \Psi_{\pm\nu} \right\}$. It is the purpose of this work to show via the methods of normal mode analysis that there exists an analogous complete set of functions that permits the solution of the associated initial value problem.

In order to facilitate notation in what follows, $g_{\pm}(z,u_z,t)$ will simply be designated as $g(z,u_z,t)$ and similarly for $E_{\pm}(z,t)$ and $B_{\pm}(z,t)$. The more complete notation will be reintroduced at the end.

Returning to the three circular mode equations, (2.101), (2.102), and (2.103), and taking a Fourier transform of the space variable such that

(2.106) $g(z,u_z,t) \sim g_k(u_z,t)e^{ikz}$

(2.107) $E(z,t) \sim E_k(t)e^{ikz}$

(2.108) $B(z,t) \sim B_k(t)e^{ikz}$

gives

(2.109) $\frac{\partial g}{\partial t} = i(\pm\Omega - ku_z)g + \frac{ne}{m} E(t)\ F(u_z)$

(2.110) $\frac{\partial E}{\partial t} = -4\pi e \int_{-\infty}^{\infty} g(u_z,t)\ du_z \mp kcB(t)$

(2.111) $\frac{\partial B}{\partial t} = \pm\ ckE(t)$

where the new functions $g(u_z,t)$, $B(t)$ and $E(t)$ each carry a k subscript. This, like the + and - subscripts that the above functions should carry, is not explicitly introduced in order to avoid the confusion that is encountered when the normal mode indices are inserted.

Rewriting the above in the matrix form and defining two new quantities, Ψ and H yields

(2.112) $\frac{\partial \Psi}{\partial t} = H\Psi$

where Ψ is a three element " state vector " defined by

$$(2.113)\quad \Psi \equiv \begin{bmatrix} g(u,t) \\ E(t) \\ B(t) \end{bmatrix}$$

and H is an integral matrix operator given by

$$(2.114)\quad H = \begin{bmatrix} i(\pm\Omega - ku_z) & \frac{ne}{m} F(u_z) & 0 \\ -4\pi e\int & 0 & \mp kc \\ 0 & \pm kc & 0 \end{bmatrix}$$

In order to find the necessary normalization coefficient and to determine the expansion coefficients $a_\alpha(\nu_\alpha, k)$ and $A_k(\nu)$, we also define an adjoint matrix equation given by

$$(2.115)\quad \frac{\partial \Psi^+}{\partial t} = \Psi^+ H^+$$

where

$$(2.116)\quad (\Psi^+, H\Psi) \equiv (\Psi^+ H^+, \Psi)$$

This last stipulation will serve as our orthogonality condition.

2.7.1 Adjoint Equation

The component elements of the matrix operator H^+ can be determined by using equation (2.116) where we define the inner product (χ^+, Ψ) as

$$(2.117)\quad (\chi^+, \Psi) \equiv \int_{-\infty}^{\infty} G^+(u_z) g(u_z)\, du_z + e^+E + b^+B$$

where χ^+ is a three element row vector given by

$$(2.118) \quad \chi^+ \equiv \left| G^+(u_z), \quad e^+, \quad b^+ \right|$$

Using these definitions and remembering that in our case,

$$(2.119) \quad \Psi^+ \equiv \left| g^+(u_z,t), \quad E^+(t), \quad B^+(t) \right|$$

yields

$$(2.120) \quad H^+ = \begin{bmatrix} i(\pm\Omega - ku_z) & \frac{ne}{m}\int F(u_z) & 0 \\ -4\pi e & 0 & \mp ck \\ 0 & \pm ck & 0 \end{bmatrix}$$

The usefulness of the above will become evident when the normal mode adjoint equations are solved.

2.7.2 The Normal Mode Solutions

Returning now to equations (2.109), (2.110), (2.111), we propose that the time dependence is of the form $e^{-ik\nu t}$ such that

$$(2.121) \quad g(u_z,t) \sim g_\nu(u_z)\, e^{-ik\nu t}$$

which implies that

$$(2.122) \quad E(t) \sim E_\nu\, e^{-ik\nu t}$$

$$(2.123) \quad B(t) \sim B_\nu\, e^{-ik\nu t}$$

where the index, ν , is a complex variable and may be continuous or discrete. Inserting these quantities into the equations (2.109), (2.110) and (2.111) yields

$$(2.124)\quad i\left[k(u_z-\nu)\mp\Omega\right] g_\nu(u_z) = \frac{ne}{m} F(u_z)E_\nu$$

$$(2.125)\quad -ik\nu E_\nu = -4\pi e \int_{-\infty}^{\infty} g_\nu(u_z)du_z \mp kcB_\nu$$

$$(2.126)\quad i\nu B_\nu = \mp c E_\nu$$

where we must keep in mind that the quantities $g_\nu(u_z)$, E_ν , and B_ν should be written $g_{\pm\,\nu,k}(u_z)$, $E_{\pm\,\nu,k}$ and $B_{\pm\,\nu,k}$, respectively.

It must now be remembered that E_ν is an expansion coefficient such that

$$(2.127)\quad E_{\pm k}(t) = \sum_\alpha a_\alpha(k,\nu_\alpha)E_{\nu_\alpha} e^{-ik\nu_\alpha t} + \\ + \int_{-\infty}^{\infty} A_k(\nu)E_\nu e^{-ik\nu t}d\nu$$

and where $B_{\pm k}(t)$ and $g_{\pm k}(u_z,t)$ are of the same form i.e.

$$(2.128)\quad B_{\pm k}(t) = \sum_\alpha a_\alpha B_{\nu_\alpha} e^{-ik\nu_\alpha t} + \int_{-\infty}^{\infty} A_k(\nu)B_\nu e^{-ik\nu t}d\nu$$

$$(2.129)\quad g_{\pm k}(u_z,t) = \sum_\alpha a_\alpha g_{\nu_\alpha}(u_z)e^{-ik\nu_\alpha t} + \\ + \int_{-\infty}^{\infty} A_k(\nu)g_\nu(u_z)e^{-ik\nu t}d\nu$$

where $a_\alpha(k, \nu_\alpha)$ and $A_k(\nu)$ are expansion coefficients whose algebraic forms will be determined. What is important to note is that the choice of either E_ν or B_ν is arbitrary since it is the product of $a_\alpha E_{\nu_\alpha}$, and $A_k(\nu)E_\nu$, and $a_\alpha B_{\nu_\alpha}$ and $A_k(\nu)B_\nu$ which will determine the quantities $g_{\pm k}(u_z,t)$, $E_{\pm k}(t)$, and $B_{\pm k}(t)$. A particular choice of E_ν alters a_α and $A_k(\nu)$ only and not the three quantities, $g_{\pm k}$, $E_{\pm k}$ and $B_{\pm k}$. Shure [Shure, 1962] resolves a similar point by simply stating that the equations (2.124), (2.125), (2.126) are linear and homogeneous in the three quantities, $g_\nu(u_z)$, E_ν, and B_ν. A convenient choice for E_ν is

$$(2.130)\quad E_\nu = \frac{4\pi e i\nu}{k}$$

This implies that our equations become

$$(2.131)\quad \left[(u_z-\nu) \mp \frac{\Omega}{k}\right] g_\nu(u_z) = \frac{w_p^2}{k^2}\nu\, F(u_z)$$

$$(2.132)\quad \int_{-\infty}^{\infty} g_\nu(u_z)\, du_z = (c^2 - \nu^2)$$

$$(2.133)\quad B_\nu = \mp \frac{4\pi c e}{k}$$

We may now consider the solutions of the above. Again the allowed solutions to (2.131) must be broken up into various classes.

Class 1

These are the solutions for complex ν values, this means that

$$(2.134)\quad g_{\nu_j}(u_z) = \frac{w_p^2}{k^2}\,\nu_j\,\frac{F(u_z)}{(u_z - \nu_j \mp \frac{\Omega}{k})}$$

where ν_j are the roots of the equation

$$(2.135)\quad \frac{w_p^2}{k^2}\,\nu_j \int \frac{F(u_z)\,du_z}{(u_z - \nu_j \mp \frac{\Omega}{k})} = (c^2 - \nu_j^2)$$

Class 2a

In this class of solutions, we allow distributions in the sense of Schwartz and let $g_\nu(u_z)$ take the form

$$(2.136)\quad g_\nu(u_z) = \frac{w_p^2}{k^2}\,\nu\, P\left\{\frac{F(u_z)}{u_z - \nu \mp \frac{\Omega}{k}}\right\} + \lambda(\nu)\,\delta(u_z - \nu \mp \frac{\Omega}{k})$$

where ν is a real and continuous variable. The symbol P indicates that the principal value of the resulting integral is to be taken and $\delta(u_z - \nu \mp \frac{\Omega}{k})$ represents the Dirac delta function. The function $\lambda(\nu) = \lambda_k(\nu)$ is determined by the auxiliary condition (2.132) i.e.

$$(2.137)\quad (c^2 - \nu^2) - \frac{w_p^2}{k^2}\,\nu\; P\int_{-\infty}^{\infty} \frac{F(u_z)\,du_z}{(u_z - \nu \mp \frac{\Omega}{k})} = \lambda(\nu)$$

Class 2b

In this case, ν is again real but $F(\nu \pm \frac{\Omega}{k}) = 0$.

Again,

$$(2.138)\quad g_{\nu}(u_z) = \frac{w_p^2}{k^2}\nu \; P\left\{\frac{F(u_z)}{u_z - \nu \mp \frac{\Omega}{k}}\right\} + \lambda(\nu)\delta(u_z - \nu \mp \frac{\Omega}{k})$$

and $\lambda(\nu)$ is determined by the auxiliary condition

$$(2.139)\quad (c^2 - \nu^2) - \frac{w_p^2}{k^2}\nu \; P\int_{-\infty}^{\infty} \frac{F(u_z)\, du_z}{(u_z - \nu \mp \frac{\Omega}{k})} = \lambda(\nu)$$

Here, the principal value sign is not needed but is indicated to distinguish this case from Class 2c.

Class 2c

Here, ν is real, $F(\nu \pm \frac{\Omega}{k}) = 0$ and $\lambda(\nu) = 0$.

Consequently, we have

$$(2.140)\quad g_{\nu_i}(u_z) = \frac{w_p^2}{k^2}\nu_i \; \frac{F(u_z)}{(u_z - \nu_i \mp \frac{\Omega}{k})}$$

and

$$(2.141)\quad (c^2 - \nu_i^2) - \frac{w_p^2}{k^2}\nu_i \int_{-\infty}^{\infty} \frac{F(u_z)\, du_z}{(u_z - \nu_i \mp \frac{\Omega}{k})} = 0$$

2.7.3 Solutions to the Adjoint Equations

Using the definitions expressed by equation (2.115) and the evaluation of H^+ given in equation (2.120), we obtain the three adjoint equations

$$(2.142) \quad \frac{\partial g^+(u_z,t)}{\partial t} = i(\pm\Omega - ku_z)g^+ - 4\pi e E^+(t)$$

$$(2.143) \quad \frac{\partial E^+}{\partial t} = \frac{ne}{m}\int_{-\infty}^{\infty} F(u_z)g^+(u_z)\, du_z \pm kcB^+$$

$$(2.144) \quad \frac{\partial B^+}{\partial t} = \mp kcE^+$$

where we should write $g^+_{\pm k}(u,t)$, $E^+_{\pm k}(t)$ and $B^+_{\pm k}(t)$. Again, we assume that

$$(2.145) \quad g^+(u_z,t) \sim g^+_\nu(u_z)e^{-ik\nu t}$$

which implies that

$$(2.146) \quad B^+(t) \sim B^+_\nu\, e^{-ik\nu t}$$

$$(2.147) \quad E^+(t) \sim E^+_\nu\, e^{-ik\nu t}$$

Inserting this into the adjoint equations (2.142), (2.143) and (2.144) gives

$$(2.148) \quad i\left[k(u_z-\nu) \mp \Omega\right]g^+_\nu = -4\pi e E^+_\nu$$

$$(2.149) \quad -ik\nu B^+_\nu = \mp kcE^+_\nu$$

$$(2.150) \quad -ik\nu E^+_\nu = \frac{ne}{m}\int_{-\infty}^{\infty} F(u_z)g^+_\nu(u_z)du_z \pm kcB^+_\nu$$

A convenient choice of E^+_ν is

$$(2.151) \quad E^+_\nu = -i\,\frac{ne\nu}{mk}$$

This reduces the adjoint equations to

$$(2.152)\quad \left(u_z - \nu \mp \frac{\Omega}{k}\right) g^+_\nu(u_z) = \frac{w_p^2}{k^2}\nu$$

$$(2.153)\quad (c^2 - \nu^2) = \int_{-\infty}^{\infty} F(u_z)\, g^+_\nu(u_z)\, du_z$$

$$(2.154)\quad B^+_\nu = \mp \frac{nec}{mk}$$

The reason for choosing E^+_ν as we did can hardly be obvious at this point. It must, however, be made clear that the choice is at least self-consistent, i.e. the adjoint equations are linear and homogeneous in the so-called field quantities: $g^+_\nu(u_z)$, E^+_ν, and B^+_ν. We must constantly bear in mind here that the only reason for considering the adjoint equations is to find a set of functions, $\{\Psi^+_\nu(u_z)\}$ (adjoint state vectors) that will be orthogonal to the physical system's state vector set, $\{\Psi_\nu(u_z)\}$, which is complete in u_z in the sense that any vector, $G(u_z, 0)$ may be expanded as a linear combination of the set, $\{\Psi_\nu(u_z)\}$. We may now concentrate on obtaining the solutions to the adjoint equations; these may also be considered as falling into various classes.

Class 1

These are the solutions for complex values of ν with a non-zero imaginary part. Here we have

$$(2.155)\quad g^+_{\nu_j}(u_z) = \frac{w_p^2}{k^2}\,\frac{\nu_j}{\left(u_z - \nu_j \mp \frac{\Omega}{k}\right)}$$

and

$$(2.156)\quad (c^2 - \nu_j^2) = \frac{w_p^2}{k^2}\,\nu_j \int_{-\infty}^{\infty} \frac{F(u_z)\,du_z}{(u_z - \nu_j \mp \frac{\Omega}{k})}$$

Class 2a

In this case, ν is real and $F(\nu \pm \frac{\Omega}{k}) \neq 0$. Considering distributions in the sense of Schwartz, we have

$$(2.157)\quad g^{+}_{\nu}(u_z) = \frac{w_p^2}{k^2}\,\nu\, P\left\{\frac{1}{u_z - \nu \mp \frac{\Omega}{k}}\right\} + \lambda^{+}(\nu)\,\delta(u_z - \nu \mp \frac{\Omega}{k})$$

where $\lambda^{+}(\nu)$ is determined by the auxiliary condition

$$(2.158)\quad (c^2 - \nu^2) - \frac{w_p^2}{k^2}\,\nu \int \frac{F(u_z)\,du_z}{(u_z - \nu \mp \frac{\Omega}{k})} = F(\nu \pm \frac{\Omega}{k})\,\lambda^{+}(\nu)$$

where we see that

$$(2.159)\quad \lambda^{+}(\nu) = \frac{\lambda(\nu)}{F(\nu \pm \frac{\Omega}{k})}$$

The above relation is a direct result of choosing $E^{+} = \frac{-ine\nu}{mk}$; it will be seen that the above is very important in the determination of the normalization coefficient for the continuous case which will in turn permit us to determine $A_k(\nu)$, the expansion coefficient for the continuous case.

Class 2b

Here, ν is real, $F(\nu \pm \frac{\Omega}{k}) = 0$ and $\lambda(\nu) \neq 0$. Here we must take $E^+_\nu = 0$. This is evident from the following. We would have

$$(2.160)\quad g^+_\nu(u_z) = \frac{w_p^2}{k^2}\nu\, P\left\{\frac{1}{u_z - \nu \mp \frac{\Omega}{k}}\right\} + \lambda^+(\nu)\delta(u_z - \nu \mp \frac{\Omega}{k})$$

and

$$(2.161)\quad (c^2 - \nu^2) = \frac{w_p^2}{k^2}\nu \int \frac{F(u_z)}{(u_z - \nu \mp \frac{\Omega}{k})}\, du_z + \lambda^+(\nu)F(\nu \pm \frac{\Omega}{k})$$

or

$$(2.162)\quad \lambda(\nu) = \lambda^+(\nu)F(\nu \pm \frac{\Omega}{k})$$

as in equation (2.159). Clearly, this can not hold, for here $F(\nu \pm \frac{\Omega}{k}) = 0$ and $\lambda(\nu) \neq 0$. A consistent choice is $E^+_\nu = 0$. This implies that

$$(2.163)\quad (u_z - \nu \mp \frac{\Omega}{k})g^+_\nu(u_z) = 0$$

$$(2.164)\quad \int_{-\infty}^{\infty} F(u_z)g^+_\nu(u_z)\, du_z = 0$$

$$(2.165)\quad B^+_\nu = 0.$$

Consistent with the above is

$$(2.166)\quad g^+_\nu(u_z) = \delta(u_z - \nu \mp \frac{\Omega}{k}).$$

The auxiliary condition then becomes

$$(2.167) \quad \int_{-\infty}^{\infty} F(u_z)\delta(u_z - \nu \mp \frac{\Omega}{k})\, du_z = F(\nu \pm \frac{\Omega}{k}) = 0$$

Thus our delta function solution leads to an identity.

Class 2c

In this case, $\nu = \nu_i$ is real and $F(\nu_i \pm \frac{\Omega}{k}) =$ $= \lambda(\nu_i) = 0$. The solution to the adjoint equation is

$$(2.168) \quad g^{+}_{\nu_i}(u_z) = \frac{w_p^2}{k^2} \frac{\nu_i}{(u_z - \nu_i \mp \frac{\Omega}{k})}$$

and the auxiliary condition gives

$$(2.169) \quad (c^2 - \nu_i^2) = \frac{w_p^2}{k^2} \nu_i \int_{-\infty}^{\infty} \frac{F(u_z)}{(u_z - \nu_i \mp \frac{\Omega}{k})} du_z$$

2.7.4 Orthogonality and Normalization Coefficients

In determining the adjoint state vector Ψ^+, we stipulated that

$$(2.116) \quad (\Psi^+, H\Psi) = (\Psi^+ H^+, \Psi).$$

But remembering that

$$(2.112) \quad H\Psi = \frac{\partial \Psi}{\partial t}$$

$$(2.115) \quad \Psi^+ H^+ = \frac{\partial \Psi^+}{\partial t}$$

and introducing the normal mode time solutions

(2.170) $\Psi \sim \Psi_{\nu} e^{-ik\nu t}$

(2.171) $\Psi^{+} \sim \Psi^{+}_{\nu'} e^{-ik\nu' t}$

we arrive at the orthogonality condition

(2.172) $(\nu - \nu')(\Psi^{+}_{\nu'}, \Psi_{\nu}) = 0$

We must note in the above equation (2.170) is simply the vector representation of equations (2.121), (2.122) and (2.123) and equation (2.171) similarly represents equations (2.145), (2.146) and (2.147).

Equation (2.172) implies that the state vectors, $\Psi^{+}_{\nu'}$ and Ψ_{ν} are orthogonal over the range of the u_z integration; in this case, $-\infty \leq u_z \leq +\infty$. Our orthogonality condition may be expressed as

(2.173) $(\Psi^{+}_{\nu'_{\alpha}}, \Psi_{\nu_{\alpha}}) = N(\nu_{\alpha})\delta_{\nu_{\alpha}\, \nu'_{\alpha}}$

for the discrete case, and symbolically, for the continuous case, as

(2.174) $(\Psi^{+}_{\nu'}, \Psi_{\nu}) = N(\nu)\delta(\nu - \nu')$

where $N(\nu_{\alpha})$ and $N(\nu)$ are the normalization coefficients. To determine the algebraic forms of $N(\nu_{\alpha})$ and $N(\nu)$, we must consider the various classes of allowed solutions. Basically, there are only two cases to consider. The solutions for real and continuous values of ν and those solutions for discrete values of ν, both real and complex.

2.7.5 Determination of the Normalization Coefficients

Turning now to equation (2.173) and introducing the discrete mode solutions for $\Psi^{+}_{\nu'}$ and Ψ_{ν}, we obtain

$$(2.175)\quad N(\nu_\alpha) = \int_{-\infty}^{\infty} g^{+}_{\nu_\alpha}\, g_{\nu_\alpha}\, du_z + E^{+}_{\nu_\alpha} E_{\nu_\alpha} + B^{+}_{\nu_\alpha} B_{\nu_\alpha}$$

$$= \left(\frac{w_p^2}{k^2}\right)^2 \nu_\alpha^2 \int_{-\infty}^{\infty} \frac{F(u_z)\, du_z}{(u_z - \nu_\alpha \mp \frac{\Omega}{k})^2} + \frac{w_p^2}{k^2}(c^2 - \nu_\alpha^2)$$

where ν_α represents the real and complex discrete roots.

Turning our attention now to the continuous case, we begin by expressing the final solution to our problem in terms of a sum of discrete normal mode solutions plus an integral over all continuous mode solutions. This means that we assume completeness and the existence of the expansion coefficients, $a_\alpha(k, \nu_\alpha)$ and $A_k(\nu)$ such that

$$(2.176)\quad \Psi_k(u_z, 0) = \sum_\alpha a_\alpha \Psi_{\nu_\alpha} + \int_{-\infty}^{\infty} A_k(\nu')\, \Psi_{\nu'}\, d\nu'.$$

Again, we must keep in mind the fact that not all subscripts are shown; actually, we should add the $\pm$, k subscripts to each symbol.

Multiplying the above equation by Ψ_{ν}^{+} and integrating the results yields

$$(2.177)\quad \int_{-\infty}^{\infty} \Psi_{\nu}^{+}\,\Psi(u_z,0)\,du_z = (\Psi_{\nu}^{+},\Psi(u_z,0)) =$$

$$= \sum_{\alpha} a_{\alpha} \int_{-\infty}^{\infty} \Psi_{\nu}^{+}\,\Psi_{\nu_{\alpha}}\,du_z +$$

$$+\int_{-\infty}^{\infty} \Psi_{\nu}^{+} \int_{-\infty}^{\infty} A_k(\nu')\Psi_{\nu'}\,d\nu'\,du_z \quad .$$

The term involving the sum is seen to be zero since the continuous adjoint state vector, Ψ_{ν}^{+}, was chosen to be orthogonal to the system's discrete state vector, $\Psi_{\nu_{\alpha}}$. It is from the second term that we shall find the normalization coefficient for the continuous case. Rewriting this second term in component form, expanding the results and introducing the proper values for $E_{\nu'}$ and $B_{\nu'}$ yields

$$(2.178)\quad (\Psi_{\nu}^{+},\Psi(u_z,0)) =$$

$$= \left(\frac{w_p^2}{k^2}\right)^2 \nu \quad P\iint \frac{\nu' F(u_z)A(\nu')\,d\nu'\,du_z}{(u_z-\nu'\mp\frac{\Omega}{k})(u_z-\nu\mp\frac{\Omega}{k})}$$

$$+\frac{w_p^2}{k^2}\,\lambda^{+}(\nu)\quad P\iint \frac{\nu' F(u_z)A(\nu')\delta(u_z-\nu\mp\frac{\Omega}{k})\,d\nu'\,du_z}{(u_z-\nu'\mp\frac{\Omega}{k})}$$

$$+ \frac{w_p^2}{k^2}\nu \, P \iint \frac{\lambda(\nu')A(\nu')\delta(u_z - \nu' \mp \frac{\Omega}{k})\, d\nu' du_z}{(u_z - \nu \mp \frac{\Omega}{k})}$$

$$+ \lambda^+(\nu) \iint \lambda(\nu')A(\nu')\delta(u_z - \nu \mp \frac{\Omega}{k})\delta(u_z - \nu' \mp \frac{\Omega}{k}) \, d\nu' \, du_z$$

$$+ \frac{w_p^2}{k^2}\nu \int_{-\infty}^{\infty} \nu' A(\nu') d\nu' + \frac{w_p^2}{k^2} c^2 \int_{-\infty}^{\infty} A(\nu') d\nu'$$

The order of integration in the above is immaterial except in the first integral. Here we are dealing with a doubly singular integral a proper treatment of which requires the use of the Poincaré-Bertrand transformation [Muskhelishvili, 1953]. Performing the delta distribution integrations gives

(2.179) $(\Psi_\nu^+, \Psi(u_z, 0)) =$

$$= \left(\frac{w_p^2}{k^2}\right)^2 \nu \; P \iint \frac{\nu' F(u_z) A(\nu') \, d\nu' \, du_z}{(u_z - \nu' \mp \frac{\Omega}{k})(u_z - \nu \mp \frac{\Omega}{k})}$$

$$+ \frac{w_p^2}{k^2} \lambda^+(\nu) F(\nu \pm \frac{\Omega}{k}) \, P \int_{-\infty}^{\infty} \frac{\nu' A(\nu')}{(\nu - \nu')} \, d\nu'$$

$$+ \frac{w_p^2}{k^2} \nu \, P \int_{-\infty}^{\infty} \frac{\lambda(\nu')A(\nu')}{(\nu' - \nu)} \, d\nu'$$

$$+ \lambda^+(\nu)\lambda(\nu)A(\nu)$$

$$+ \frac{w_p^2}{k^2} \nu \int_{-\infty}^{\infty} \nu' A(\nu') d\nu' + \frac{w_p^2}{k^2} c^2 \int_{-\infty}^{\infty} A(\nu')\, d\nu' .$$

It is shown in Appendix A how the use of the Poincaré-Bertrand transformation and the method of partial fractions reduces the above to

$$(2.180) \quad (\Psi_\nu^+, \Psi(u_z, 0)) = N(\nu) A(\nu)$$

where $N(\nu)$ is shown to be given by

$$(2.181) \quad N(\nu) = \frac{\left(\frac{w_p^2}{k^2}\right)^2 \pi^2 \nu^2 F^2(\nu \pm \frac{\Omega}{k}) + \lambda^2(\nu)}{F(\nu \pm \frac{\Omega}{k})} .$$

2.7.6 Expansion Coefficients

Assuming as we did for the determination of $N(\nu_\alpha)$ and $N(\nu)$ that the functions $\left\{\Psi_{\nu_\alpha}, \Psi_\nu\right\}$ form a complete set, we may express the total solutions as a sum of discrete normal mode solutions plus an integral over all continuous normal mode solutions, i.e.

$$(2.182) \quad \Psi(u_z, 0) = \sum_\alpha a_\alpha \Psi_{\nu_\alpha} + \int_{-\infty}^{\infty} A_k(\nu') \Psi_{\nu'} d\nu' .$$

Consequently, the discrete mode coefficient is given by

$$(2.183) \quad a_\alpha(k, \nu_\alpha) = \frac{(\Psi_{\nu_\alpha}^+, \Psi(u_z, 0))}{N(\nu_\alpha)}$$

where

$$(2.184)\quad \Psi(u_z,0) = \Psi_k(u_z,0) = \begin{bmatrix} g_k(u_z,0) \\ E_k(0) \\ B_k(0) \end{bmatrix}$$

Therefore,

$$(2.185)\quad (\Psi^+_{\nu_\alpha}, \Psi(u_z,0)) =$$

$$= \int_{-\infty}^{\infty} g^+_{\nu_\alpha}(u_z) g_k(u_z,0) du_z + E^+_{\nu_\alpha} E_k(0) + B^+_{\nu_\alpha} B_k(0)$$

$$= \frac{w_p^2}{k^2} \nu_\alpha \int_{-\infty}^{\infty} \frac{g_k(u_z,0)}{(u_z - \nu_\alpha \mp \frac{\Omega}{k})} du_z + \frac{ne\,\nu_\alpha}{mik} (E_k(0) \mp iB_k(0)).$$

The continuous mode expansion coefficient may be obtained in a similar manner. We see from the above that

$$(2.186)\quad A_k(\nu) = \frac{1}{N(\nu)} \int_{-\infty}^{\infty} \Psi^+_\nu \Psi(u_z,0)\, du_z$$

where

$$(2.187)\quad \int_{-\infty}^{\infty} \Psi^+_\nu \Psi(u_z,0)\, du_z =$$

$$= \int_{-\infty}^{\infty} g^+_\nu (u_z) g_k(u_z,0) du_z + E^+_\nu E_k(0) + B^+_\nu B_k(0)$$

$$= \frac{w_p^2}{k^2} \nu\, P \int \frac{g_k(u_z,0)}{(u_z - \nu \mp \frac{\Omega}{k})} du_z + \lambda^+(\nu) g_k(\nu,0) +$$

$$+ \frac{ne\,\nu}{mik} (E_k(0) \mp iB_k(0)).$$

Therefore, we may write

$$(2.188)\quad A(\nu) = \frac{\lambda(\nu)g_k(\nu,0)+F(\nu\pm\frac{\Omega}{k})\left[\frac{w_p^2}{k^2}\nu\, P\int\frac{g_k(u_z,0)}{(u_z-\nu\mp\frac{\Omega}{k})}\,du_z+\frac{ne\nu}{mik}(E_k(0)\mp iB_k(0))\right]}{\left(\frac{w_p^2}{k^2}\right)^2\pi^2\nu^2F^2(\nu\pm\frac{\Omega}{k})+\lambda^2(\nu)}$$

The above certainly holds for ν in class 2a. It also holds for ν belonging to class 2b since then, $F(\nu\pm\frac{\Omega}{k}) = 0$ and

$$(2.189)\quad A(\nu) = \frac{g_k(\nu,0)}{\lambda(\nu)}.$$

<u>2.7.7 Final Solution of the Transverse Oscillation Problem</u>

This then completes the treatment of the initial value problem for the transverse modes. The three field quantities are given by

$$(2.190)\quad g_k(u_z,t) = \sum_\alpha a_\alpha(k,\nu_\alpha)g_{\nu_\alpha}e^{-ik\nu_\alpha t} + \int_{-\infty}^{\infty}A_k(\nu)g_\nu(u_z)e^{-ik\nu t}\,d\nu$$

$$(2.191)\quad E_k(t) = \sum_\alpha a_\alpha(k,\nu_\alpha)E_{\nu_\alpha}e^{-ik\nu_\alpha t} + \int_{-\infty}^{\infty}A_k(\nu)E_\nu\, e^{-ik\nu t}\,d\nu$$

$$(2.192)\ B_k(t) = \sum_{\alpha} a_\alpha(k, \nu_\alpha) B_{\nu_\alpha} e^{-ik\nu_\alpha t} + $$

$$+ \int_{-\infty}^{\infty} A_k(\nu) B_\nu \, e^{-ik\nu t} d\nu$$

where all quantities tacitly carry the $\pm$ signs designating whether we are considering the right or left polarized modes. The measurable field quantities are given by taking the inverse space Fourier transform of the above quantities, i.e.

$$(2.193)\ g_{\pm}(z, u_z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_{\pm,k}(u_z, t) e^{-ikz} dk$$

$$(2.194)\ E_{\pm}(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{\pm,k}(t) e^{-ikz}\, dk$$

$$(2.195)\ B_{\pm}(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B_{\pm,k}(t) e^{-ikz}\, dk$$

It must, however, be stressed here that the completemess of our set of solutions, $\{\Psi_\nu\}$, is contingent on the existence of the coefficients $a_\alpha(k, \nu_\alpha)$ and $A_k(\nu)$ which was assumed. A justification of this requires the use of the methods of singular integral equations in the solution of an inhomogeneous Hilbert problem. This rather involved demonstration is considered outside the scope of the present endeavor.

III APPLICATIONS

In this chapter, we shall demonstrate the use of the previously derived general results by applying them to a few particular cases. This should bring to light the difficulties encountered in dealing with even the simplest problems.

3.1 An Initially Isotropic Perturbed Distribution

Let us consider the transverse and longitudinal mode solutions for the case where the perturbed distribution function is initially isotropic in velocities, i.e.

$$(3.1) \quad f(z,\bar{u},0) = f(z,|\bar{u}|,0).$$

Referring to equations (2.25) and (2.26) yields for the transverse modes

$$(3.2) \quad g_x(z,u_z,0) = 0$$

$$(3.3) \quad g_y(z,u_z,0) = 0.$$

Consequently, the initial values of the spatial Fourier transform of equations (3.2) and (3.3) are also zero. If these results coupled to the assumption that there exists no field quantities at times, $t \leq 0$ are inserted into equations (2.185) and (2.188), we obtain for the transverse case

(3.4) $\quad a_\alpha(\nu_\alpha, k) = 0$

(3.5) $\quad A_k(\nu) = 0.$

This then indicates that

(3.6) $\quad g_x(z, u_z, t) = g_y(z, u_z, t) = 0$

(3.7) $\quad E_x(z, t) = E_y(z, t) = 0$

(3.8) $\quad B_x(z, t) = B_y(z, t) = 0.$

Turning to the longitudinal case and considering equation (2.27), we obtain

(3.9) $\quad g_z(z, u_z, 0) \neq 0.$

Taking the spatial Fourier transform of $g_z(z, u_z, 0)$ and inserting the results into equations (2.89) and (2.90) yields the expansion coefficients, $a_\alpha(\nu_\alpha, k)$ and $A_k(\nu)$, for the longitudinal case. Applying the latter into equations (2.91) and (2.94) and inserting these results into (2.96) and (2.97) yields $g_z(z, u_z, t)$ and $\varphi(z, t)$.

We may summarize by saying that a perturbed distribution that is initially isotropic in velocities gives rise to the longitudinal mode but no transverse phenomena.

We might also add that it is possible by a proper choice of $f(z, \bar{u}, 0)$ to have transverse oscillations with no accompanying longitudinal mode.

3.2 Two Cases of Monodirectional Initial Perturbations

To further illustrate the use of our methods and to indicate the uncoupled nature of the transverse and longitudinal modes, we shall consider some monodirectional initial distributions. Suppose that, initially, all particles moved along the z direction such that

$$(3.10) \quad f(z,\bar{u},0) = h(z,u_z)\delta(u_y)\delta(u_x) \ .$$

Inserting the above into equations (2.25), (2.26), (2.27) yields

$$(3.11) \quad g_x(z,u_z,0) = 0$$

$$(3.12) \quad g_y(z,u_z,0) = 0$$

$$(3.13) \quad g_z(z,u_z,0) = u_z h(z,u_z).$$

The Fourier transforms of $g_x(z,u_z,0)$ and $g_y(z,u_z,0)$ are similarly zero and consequently equations (2.185) and (2.188) yield for the transverse case

$$(3.14) \quad a_\alpha(\nu_\alpha,k) = 0$$

$$(3.15) \quad A_k(\nu) = 0$$

where we assumed that

(3.16) $B_{\pm,k}(0) = E_{\pm,k}(0) = 0.$

Consequently, the results expressed in equations (3.6), (3.7) and (3.8) hold here also.

Considering now the longitudinal case and specifically equations (2.89) and (2.90), we have for the longitudinal case

(3.17) $a_\alpha(\nu_\alpha, k) \neq 0$

(3.18) $A_k(\nu) \neq 0.$

Consequently, we may find $g_z(z,u_z,t)$ and $\varphi(z,t)$.

The above results indicate that electrons initially moving along the oscillation direction, z, will continue to do so for times, $t > 0$ and will not give rise to transverse phenomena.

To complete our present considerations, we shall now examine the case where electrons are initially made to travel in a direction perpendicular to the direction of longitudinal oscillations. Suppose that

(3.19) $f(z,\bar{u},0) = h(z,u_x)\delta(u_y)\delta(u_z).$

Applying (3.19) to equations (2.25), (2.26) and (2.27) yields

(3.20) $g_x(z,u_z,0) = \delta(u_z) \int_{-\infty}^{\infty} u_x \, h(z,u_x) \, du_x$

(3.21) $g_y(z,u_z,0) = 0$

$$g_z(z,u_z,0) = u_z\delta(u_z) \int h(z,u_x)\, du_x = 0. \tag{3.22}$$

Taking the spatial Fourier transform of equations (3.20), (3.21) and (3.22) and applying the results in the determination of the expansion coefficients as was done previously, indicates the existence of the transverse modes and the absence of longitudinal phenomena. Therefore, we conclude that electrons whose initial direction of motion is perpendicular to the direction of longitudinal oscillations will continue in this perpendicular motion for $t > 0$ and will not give rise to longitudinal effects.

3.3 An Example Problem (Longitudinal Mode)

In this section, we shall consider one particular problem illustrating the solution of the longitudinal mode of oscillation. Consider the following total distribution function

$$F(\bar{r},\bar{u},t) = n_o F_o(|\bar{u}|) + f(z,\bar{u},t) \tag{3.23}$$

where $n_o F_o$ is a Maxwell-Boltzmann distribution such that

$$n_o F_o = n_o\left(\frac{\beta}{\pi}\right)\sqrt{\frac{\beta}{\pi}}\; e^{-\beta u^2} \tag{3.24}$$

$$\beta = \frac{m}{2KT} \tag{3.25}$$

$$(3.26) \quad u^2 = u_x^2 + u_y^2 + u_z^2$$

and n_o is the electron density at equilibrium. We next let

$$(3.27) \quad f(z,\bar{u},0) = h(z,u_x,u_y,0)\, e^{-\beta u_z^2}.$$

Although any perturbed distribution satisfying the condition that

$$(3.28) \quad n_o F_o \gg f(z,\bar{u},t)$$

would be valid, the above was chosen because of the mathematical simplicity that it introduces into the resulting equations.

Using equation (2.51), we have

$$(3.29) \quad u_z \bar{g}_k(u_z,0) = \iint u_z h_k(u_x,u_y,0) e^{-\beta u_z^2}\, du_x\, du_y$$

$$= u_z\, H_k(0)\, e^{-\beta u_z^2}$$

where $H_k(u_x,u_y,0)$ is the Fourier transform of $h(z,u_x,u_y,0)$ and

$$(3.30) \quad H_k(0) = \iint H_k(u_x,u_y,0)\, du_x\, du_y$$

We are now in a position to determine $A_k(\nu)$ and $a_\alpha(\nu_\alpha,k)$. Referring to equation (2.90) and inserting the proper value for $g^+_\nu(u_z)$, we obtain

$$(3.31)\quad A_k(\nu) = \frac{\lambda(\nu)\bar{g}_k(\nu,0) + \frac{w_p^2}{k^2} F'(\nu) P \int \frac{\bar{g}_k(u_z,0)\, du_z}{(u_z - \nu)}}{\lambda^2(\nu) + \left[\pi \frac{w_p^2}{k^2} F'(\nu)\right]^2}$$

where $\lambda(\nu)$ is given by equation (2.67), i.e.

$$(3.32)\quad \lambda(\nu) = 1 - \frac{w_p^2}{k^2} P \int \frac{F'(u_z)\, du_z}{(u_z - \nu)}$$

Similarly, referring to equation (2.89) and inserting the proper value for $g^+_{\nu_\alpha}(u_z)$ gives

$$(3.33)\quad a_\alpha = \frac{\int_{-\infty}^{\infty} \frac{\bar{g}_k(u_z,0)}{(u_z - \nu_\alpha)}\, du_z}{\frac{w_p^2}{k^2} \int_{-\infty}^{\infty} \frac{F'(u_z)\, du_z}{(u_z - \nu_\alpha)^2}}$$

The evaluation of the above integrals requires some straight-forward but rather tedious algebra. For the sake of continuity, this tedium will be relegated to the various appendices as indicated. We have

$$(3.34)\quad F(u_z) = \left(\frac{\beta}{\pi}\right)^{3/2} \iint e^{-\beta u^2}\, du_x\, du_y = \sqrt{\frac{\beta}{\pi}}\; e^{-\beta u_z^2}$$

$$(3.35)\quad \lambda(\nu) = 1 - \frac{w_p^2}{k^2} P \int \frac{\left[-2\beta \sqrt{\frac{\beta}{\pi}}\; u_z\, e^{-\beta u_z}\right] du_z}{(u_z - \nu)}$$

$$= 1 + \frac{2w_p^2}{k^2}\; \beta(1 - 2\beta\, \nu^2 \varphi e^{-\beta \nu^2})$$

(Appendix B)

where by definition,

$$(3.36) \quad \varphi(\beta,\nu) = \sum_{n=0}^{\infty} \frac{\beta^n \nu^{2n}}{(2n+1)n!}$$

$$(3.37) \quad P\int_{-\infty}^{\infty} \frac{\bar{g}_k(u_z,0)}{(u_z-\nu)}\,du_z = H_k(0)\int_{-\infty}^{\infty} \frac{e^{-\beta u_z^2}du_z}{(u_z-\nu)}$$

$$= H_k(0)\left[-2\beta\sqrt{\frac{\pi}{\beta}}\,\nu\,\varphi e^{-\beta\nu^2}\right]$$

(Appendix B)

$$(3.38) \quad \int_{-\infty}^{\infty} \frac{\bar{g}_k(u_z,0)}{(u_z-\nu_\alpha)}du_z = H_k(0)\int_{-\infty}^{\infty} \frac{e^{-\beta u_z^2}\,du_z}{(u_z-\nu_\alpha)}$$

$$= H_k(0)\sqrt{\pi}Z(\sqrt{\beta}\,\nu_\alpha)$$

where $Z(\sqrt{\beta}\,\nu_\alpha)$ is the plasma dispersion function and ν_α is a discrete root.

(Appendix B)

$$(3.39) \quad \int \frac{\left[-\frac{w_p^2}{k^2}F'(u_z)\right]}{(u_z-\nu_\alpha)^2}du_z = \frac{2w_p^2}{k^2}\beta\sqrt{\frac{\beta}{\pi}}\int \frac{u_z e^{-\beta u_z^2}}{(u_z-\nu_\alpha)^2}du_z$$

$$= \frac{2w_p^2}{k^2}\beta\sqrt{\frac{\beta}{\pi}}\left[-2\beta\sqrt{\frac{\pi}{\beta}}\,\nu_\alpha + (1-2\beta\nu_\alpha^2)\sqrt{\pi}Z(\sqrt{\beta}\,\nu_\alpha)\right]$$

(Appendix C)

Having determined these integrals, we may write out the necessary expansion coefficients. Consider the continuous case first. Reducing the function to its simplest form gives

(3.40)
$$A_k(\nu) = \frac{H_k(0)\left(1+\frac{2w_p^2}{k^2}\beta\right)e^{-\beta\nu^2}}{\left[1+\frac{2w_p^2}{k^2}\beta\left(1-2\beta\nu^2\varphi e^{-\beta\nu^2}\right)\right]^2+\left[\frac{2\pi w_p^2}{k^2}\beta\sqrt{\frac{\beta}{\pi}}\,\nu e^{-\beta\nu^2}\right]^2}$$

The coefficient for the discrete case is

$$(3.41)\quad a_\alpha(k) = \frac{H_k(0)\sqrt{\pi}\;Z(\sqrt{\beta}\,\nu_\alpha)}{\frac{2w_p^2}{k^2}\beta\sqrt{\frac{\beta}{\pi}}\left[2\beta\sqrt{\frac{\pi}{\beta}}\,\nu_\alpha-(1-2\beta\nu_\alpha^2)\sqrt{\pi}Z(\sqrt{\beta}\,\nu_\alpha)\right]}$$

Let us discuss at this point the mathematical nature of these results. We see that for a finite k

$$(3.42)\quad \lim_{\nu\to 0} A_k(\nu) = \frac{H_k(0)}{\left(1+\frac{2w_p^2}{k^2}\beta\right)}$$

$$(3.43)\quad \lim_{\nu\to\infty} A_k(\nu) = 0$$

where the characteristics of the function

$$(3.44)\quad \Psi(\beta,\nu) \equiv 2\beta\nu^2\varphi(\beta,\nu)\,e^{-\beta\nu^2}$$

have been used, i.e.

$$(3.45)\quad \lim_{\nu \to 0} \Psi(\beta,\nu) = 0$$

$$(3.46)\quad \lim_{\nu \to \infty} \Psi(\beta,\nu) = 1$$

(Appendix D)

We also note that

$$(3.47)\quad A_k(\nu) = A_k(-\nu).$$

The above characteristics and the fact that there exist no singularities in the function $A_k(\nu)$ indicate that $A_k(\nu)$ is a fairlywell behaved function of ν.

Finally we must devote some attention to the evaluation of the roots to the so called dispersion relation:

$$(3.48)\quad 0 = 1 + \int_{-\infty}^{\infty} \frac{\frac{-w_p^2}{k^2} F'(u_z)}{(u_z - \nu_\alpha)} \, du_z.$$

It is shown in Appendix E that, for the case of a Maxwell-Boltzmann equilibrium distribution, there exist no roots to the above equation of the type

$$(3.49)\quad \mathrm{Im}(\nu_\alpha) \geq 0.$$

Restricting attention therefore to the case where $\mathrm{Im}(\nu_\alpha) < 0$, we get

$$(3.50) \quad 0 = 1 + \frac{2w_p^2}{k^2}\,\beta\sqrt{\frac{\beta}{\pi}}\left[\sqrt{\frac{\pi}{\beta}} + \sqrt{\pi}\,\nu_\alpha\, Z(\sqrt{\beta}\,\nu_\alpha)\right] .$$

Separating the above into the real and imaginary parts and solving for $Re(\nu_\alpha)$ and $Im(\nu_\alpha)$ would give

$$(3.51) \quad Re(\nu_\alpha) = F_1(w_p, \beta, k)$$

and

$$(3.52) \quad Im(\nu_\alpha) = F_2(w_p, \beta, k).$$

Unfortunately, due to the complexity of $Z(\sqrt{\beta}\,\nu_\alpha)$, the above proceedure is extremely difficult; a case where it can be done is that where $Re(\nu_\alpha) \gg Im(\nu_\alpha)$. Here, we obtain

$$(3.53) \quad w_o = w_p(1 + 3/2\, k^2 \lambda_D^2)$$

and

$$(3.54) \quad k\varepsilon \simeq \sqrt{\frac{\pi}{8}}\, w_p \frac{1}{(k\,\lambda_D)^3}\, e^{-\frac{1}{2(k\,\lambda_D)^2}}$$

where by definition

$$(3.55) \quad k\,\nu_\alpha = w_o - ik\varepsilon.$$

This is the Landau dispersion relation, where

$$(3.56)\quad \lambda_D = \sqrt{\frac{KT}{4\pi ne^2}}\ , \quad \text{Debye length.}$$

The problem has therefore been reduced to finding the inverse spatial transforms of

$$(3.57)\quad \bar{g}_k(u_z,t) = \sum_\alpha a_\alpha e^{-ik\nu_\alpha t} g_{\nu_\alpha}(u_z) +$$

$$+ \int_{-\infty}^{\infty} A_k(\nu)e^{-ik\nu t} g_\nu(u_z)\, d\nu$$

and

$$(3.58)\quad \varphi_k(t) = \frac{4\pi e}{k^2}\left[\sum_\alpha a_\alpha e^{-ik\nu_\alpha t} + \int A_k(\nu)e^{-ik\nu t} d\nu\right].$$

Returning now to our "current" definition, we multiply $\bar{g}_k(u_z,t)$ by u_z. Taking the inverse transform of these results, we get

$$(3.59)\quad g_z(z,u_z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_z\bar{g}_k(u_z,t)e^{-ikz}\, dk$$

and

$$(3.60)\quad \varphi(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_k(t)\, e^{-ikz}\, dk.$$

Due to the complexity of the expansion coefficients a_α and $A_k(\nu)$ and the difficulty in determining the discrete roots for the general case, further analysis of this particular problem would be relegated to computer techniques.

IV CONCLUSIONS

The properties of a plasma immersed in a constant external magnetic field may be determined, to a first order approximation, by the solution of a modified first moment of a linearized transport equation coupled to the four Maxwell relations. For the case of plane symmetry, we can express the plasma phenomena in terms of transverse and longitudinal modes of oscillation, i.e. phenomena due to electron motion perpendicular to or parallel with the direction of the outside impressed magnetic field.

It is at this point that the equations become amenable to normal mode analysis either as a boundary value or an initial value problem. Here, we were concerned only with the latter. Considering the transverse and longitudinal modes separately and ascribing a so-called "state vector" to each, we Fourier transformed the space variable and assumed that all time quantities varied as $e^{-ik\nu t}$. We then found two sets of functions, one characteristic of the transverse mode and the other of the longitudinal mode, with the properties that each set is complete in the longitudinal velocity variable. We demonstrated a prescription for determining the expansion coefficients characteristic of each set. Using these formal relations we then could express the physical state vectors (one for the transverse mode and the other for the longitudinal mode) of an initial value problem as a linear combination of the normal mode state vectors comprising the sets.

The existence of the expansion coefficients for the longitudinal case was demonstrated by Case [Case, 1959] whereas that for the transverse mode was here assumed. The justification of the latter is rather involved for it requires the methods of singular integral equations in the solutionof an inhomogeneous Hilbert problem. Our belief in the existence of such coefficients is strengthened, however, by Shure's [Shure, 1962] demonstration of existence of similar coefficients for the transverse mode boundary value problem.

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APPENDIX A

In this section, we shall indicate how equation (2.179) yields the normalization coefficient for the continuous case. We begin by considering the evaluation of the integral involving a double singularity, I_o, where

$$(A.1) \qquad I_o \equiv P \iint_{-\infty}^{\infty} \frac{\nu' F(u_z) A(\nu') \, d\nu' \, du_z}{(u_z - \nu' \mp \frac{\Omega}{k})(u_z - \nu \mp \frac{\Omega}{k})} .$$

The order of integration may not be arbitrarily changed in the above. It can be shown by using the Bertrand-Poincare transformation [Muskhelishvili, 1953] that

$$(A.2) \qquad \int_L dt \int_L \frac{\varphi(t,t_1)\, dt_1}{(t-t_o)(t_1-t)} = -\pi^2 \varphi(t_o,t_o) + \int_L dt_1 \int_L \frac{\varphi(t,t_1) dt}{(t-t_o)(t_1-t)}$$

where $\varphi(t,t_1)$ must satisfy a Hölder condition. Assuming that our function, $\nu' F(u_z) A(\nu')$, satisfies a Hölder condition and letting

$$(A.3) \qquad x \equiv (u_z \mp \frac{\Omega}{k})$$

where we obtain

$$(A.4) \qquad -I_o = P \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{\nu' F(x \pm \frac{\Omega}{k}) \, A(\nu') \, d\nu'}{(x - \nu)(\nu' - x)} .$$

Making use of the Poincaré-Bertrand transformation and returning to the u_z variable, we get

$$(A.5) \qquad I_o = \pi^2 \nu F(\nu \pm \frac{\Omega}{k}) A(\nu) + \int d\nu' \int \frac{\nu' F(u_z) A(\nu') du_z}{(u - \nu' \mp \frac{\Omega}{k})(u_z - \nu \mp \frac{\Omega}{k})} .$$

This may be further reduced by the use of partial fractions. It becomes

$$(A.6) \qquad I_o = \pi^2 \nu F(\nu \pm \frac{\Omega}{k}) A(\nu) + P \int_{-\infty}^{\infty} \frac{\nu' A(\nu')}{(\nu - \nu')} \left[c^2 - \nu^2 - \lambda(\nu) \right] \frac{k^2}{w_p^2} (\frac{1}{\nu})$$

$$- P \int \frac{\nu' A(\nu')}{(\nu - \nu')} \left[c^2 - \nu'^2 - \lambda(\nu') \right] \frac{k^2}{w_p^2} (\frac{1}{\nu'})$$

where we used the expression

$$(A.7) \qquad c^2 - \nu^2 - \frac{w_p^2}{k^2} \nu \int \frac{F(u_z)}{(u_z - \nu \mp \frac{\Omega}{k})} du_z = \lambda(\nu) .$$

Inserting these results into (2.179) and remembering that

$$(A.8) \qquad \lambda^+(\nu) F(\nu \pm \frac{\Omega}{k}) = \lambda(\nu)$$

yields

$$(A.9) \qquad (\Psi_\nu^+ , \Psi(u_z, 0)) = A(\nu) \left[\left(\frac{w_p^2}{k^2} \right)^2 \pi^2 \nu^2 F(\nu \pm \frac{\Omega}{k}) + \right.$$

$$\left. + \frac{\lambda^2(\nu)}{F(\nu \pm \frac{\Omega}{k})} \right]$$

Defining

$$(\text{A.10}) \quad (\Psi_{\nu}^{+}, \Psi(u_z, 0)) = \delta(\nu - \nu')N(\nu)A(\nu)$$

we obtain

$$(\text{A.11}) \quad N(\nu) = \frac{\left(\frac{w_p^2}{k^2}\right)^2 \pi^2 \nu^2 F^2(\nu \pm \frac{\Omega}{k}) + \lambda^2(\nu)}{F(\nu \pm \frac{\Omega}{k})}$$

APPENDIX B

Two necessary integrals are developed in this section,

$$(B.1) \qquad I_1 \equiv P\int_{-\infty}^{\infty} \frac{ue^{-\beta u^2}}{(u-\nu)} du$$

and

$$(B.2) \qquad I_2 \equiv P\int_{-\infty}^{\infty} \frac{e^{-\beta u^2}}{(u-\nu)} du$$

where ν is a real and continuous variable. Consider I_1; adding and subtracting ν to the numerator gives

$$(B.3) \qquad I_1 = \sqrt{\frac{\pi}{\beta}} + \nu P\int_{-\infty}^{\infty} \frac{e^{-\beta u^2}}{(u-\nu)} du$$

$$(B.4) \qquad = \sqrt{\frac{\pi}{\beta}} + \nu I_2 .$$

Turning to I_2 and letting $(u-\nu) = y$ yields

$$(B.5) \qquad I_2 = e^{-\beta\nu^2} G(\beta,\nu)$$

where, by definition,

$$(B.6) \qquad G(\beta,\nu) \equiv P\int_{-\infty}^{\infty} \frac{e^{-\beta(y^2+2y\nu)}}{y} dy \quad .$$

Therefore

$$(B.7) \qquad \frac{\partial G}{\partial \nu} = -2\beta \int_{-\infty}^{\infty} e^{-\beta(y^2+2y\nu)} dy$$

where the principal value sign is no longer needed. Consequently,

$$\text{(B.8)} \quad \frac{\partial G}{\partial \nu} = -2\beta\sqrt{\frac{\pi}{\beta}}\; e^{\beta\nu^2} .$$

Assuming an infinite series solution of the type

$$\text{(B.9)} \quad G(\beta,\nu) = \sum_{n=0}^{\infty} a_n \nu^{2n+m}$$

we readily see that

$$\text{(B.10)} \quad m = 1$$

$$\text{(B.11)} \quad a_n = -2\beta\sqrt{\frac{\pi}{\beta}}\; \frac{\beta^n}{(2n+1)n!} .$$

Therefore

$$\text{(B.12)} \quad G(\beta,\nu) = -2\beta\sqrt{\frac{\pi}{\beta}}\; \nu\varphi(\beta,\nu)$$

where, by definition,

$$\text{(B.13)} \quad \varphi(\beta,\nu) \equiv \sum_{n=0}^{\infty} \frac{\beta^n \nu^{2n}}{(2n+1)n!} .$$

A simple term by term comparison of $\varphi(\beta,\nu)$ with $e^{\beta\nu^2}$ indicates that

$$\text{(B.14)} \quad \varphi(\beta,\nu) \leq e^{\beta\nu^2} .$$

When these results are introduced into I_1 and I_2, we obtain

$$(B.15)\quad I_1 = \sqrt{\frac{\pi}{\beta}}\ (1 - 2\beta\nu^2\varphi(\beta,\nu)\ e^{-\beta\nu^2})$$

$$(B.16)\quad I_2 = -2\beta\sqrt{\frac{\pi}{\beta}}\,\nu\, e^{-\beta\nu^2}\ \varphi(\beta,\nu).$$

The function, I_2, is quite important in the acoustics of rarefied gases and in several engineering problems; it is related to the plasma dispersion function which is essentially the Hilbert transform of a Gaussian. Fried and Conte [Fried and Conte, 1961] define this function as

$$(B.17)\quad Z(\tau) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{e^{-x^2}}{(x-\tau)}\, dx$$

for $\operatorname{Im}\{\tau\} > 0$ and as the analytic continuation of this for $\operatorname{Im}\{\tau\} \leq 0$.

It must be noticed that the above evaluation of

$$(B.18)\quad I_2(\beta,\nu) = P\int_{-\infty}^{\infty} \frac{e^{-\beta u^2}\, du}{(u-\nu)}$$

is valid only for ν = real value. It would be incorrect to state that the integral

$$(B.19)\quad I_2(\beta,\nu_\alpha) = \int_{-\infty}^{\infty} \frac{e^{-\beta u^2}}{(u-\nu_\alpha)}\, du$$

with ν_α = complex quantity is given by

$$(B.20)\quad -2\beta\sqrt{\frac{\pi}{\beta}}\ \nu_\alpha\, e^{-\beta\nu_\alpha^2}\ \varphi(\beta,\nu_\alpha).$$

Proof of the above statement follows. Consider the limiting case of $I_2(\nu_\alpha)$ for $\mathrm{Im}(\nu_\alpha) \rightarrow 0$.

$$\text{(B.21)} \quad \lim_{\mathrm{Im}\{\nu_\alpha\} \rightarrow 0} I_2(\nu_\alpha) = \lim_{\mathrm{Im}\{\nu_\alpha\} \rightarrow 0} \int_{-\infty}^{\infty} \frac{e^{-\beta u^2} du}{(u - \nu_0) \mp i\varepsilon}$$

where

$$\text{(B.22)} \quad \nu_\alpha \equiv \nu_0 \pm i\varepsilon .$$

Therefore, using the fact that

$$\text{(B.23)} \quad \lim_{p \rightarrow 0} \frac{p}{y^2 + p^2} = \pi \, \delta(y)$$

gives

$$\text{(B.24)} \quad \lim_{\mathrm{Im}\{\nu_\alpha\} \rightarrow 0} I_2(\nu_\alpha) = P \int_{-\infty}^{\infty} \frac{e^{-\beta u^2} du}{(u - \nu_0)} \pm i\pi e^{-\beta \nu_0^2}$$

which proves the above statement. Since the integral (B.18) is of such importance in plasma physics and other fields of science some of its mathematical characteristics will be developed in Appendix D. The integral $I_2(\beta, \nu_\alpha)$ with ν_α = complex quantity can, however, be related to the plasma dispersion function by letting $\sqrt{\beta}\, u = x$; this yields

$$\text{(B.25)} \quad I_2(\beta, \nu_\alpha) = \int_{-\infty}^{\infty} \frac{e^{-x^2} dx}{(x - \sqrt{\beta}\, \nu_\alpha)} = \sqrt{\pi}\, Z(\sqrt{\beta}\, \nu_\alpha)$$

where $Z(\sqrt{\beta}, \nu_\alpha)$ is the plasma dispersion function whose algebraic form is given by

$$(B.26)\quad Z(\tau) = i\sqrt{\pi}\, e^{-\tau^2} - 2\tau\left[1 - 2\frac{\tau^2}{3} + \frac{4\tau^4}{15} - \frac{8\tau^6}{105} + \dots\right]$$

It will be shown in Appendix E that the discrete root, ν_α, is such that

$$\mathrm{Im}(\nu_\alpha) < 0.$$

Finally, to complete the relationship between the function $\varphi(\beta,\nu)$ for ν = real quantity and the plasma dispersion function, we must remember that

$$(B.27)\quad \varphi(\beta,\nu) = \frac{1}{\nu} \int_{-\infty}^{\infty} e^{\beta x^2}\, dx.$$

APPENDIX C

Consider the integral

$$(C.1) \qquad I(\nu_\alpha) = \int_{-\infty}^{\infty} \frac{ue^{-\beta u^2}}{(u - \nu_\alpha)^2}\, du$$

where ν_α is a root of the equation

$$(C.2) \qquad 0 = 1 + \frac{2w_p^2}{k^2}\, \beta \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{\infty} \frac{ue^{-\beta u^2}}{(u - \nu_\alpha)}\, du.$$

It is shown in Appendix E that the root to the above equation cannot be real and that $\mathrm{Im}(\nu_\alpha) < 0$. We then assume these charactertics for the present. Letting

$$(C.3) \qquad dV = \frac{du}{(u - \nu_\alpha)^2} \qquad ; \qquad U = ue^{-\beta u^2}$$

and dropping the integrated term gives

$$(C.4) \qquad I(\beta, \nu_\alpha) = \int_{-\infty}^{\infty} \frac{e^{-\beta u^2}\, du}{(u - \nu_\alpha)} - 2\beta \int_{-\infty}^{\infty} \frac{u^2\, e^{-\beta u^2}}{(u - \nu_\alpha)}\, du.$$

Adding and subtracting ν_α^2 to the second term, integrating the results where possible and rearranging the terms yields

$$(C.5) \qquad I(\beta, \nu_\alpha) = -2\beta \sqrt{\frac{\pi}{\beta}}\; \nu_\alpha + (1 - 2\beta\nu_\alpha^2) \int_{-\infty}^{\infty} \frac{e^{-\beta u^2}\, du}{(u - \nu_\alpha)}.$$

Referring now to the results of Appendix B, we write

$$(C.6) \qquad I(\beta,\nu) = -2\beta\sqrt{\frac{\pi}{\beta}}\ \nu_\alpha + (1-2\beta\,\nu_\alpha^2)\sqrt{\Omega}\ Z(\beta\,\nu_\alpha)$$

where $Im(\nu_\alpha) < 0$.

APPENDIX D

Due to the importance of the plasma dispersion function and its relation to our integral (B.2), we shall devote some space here in illustrating a few of the mathematical properties that these functions have in common. From Appendix B, we have

(D.1) $$I(\beta, \nu) = P \int \frac{ue^{-\beta u^2}}{(u-\nu)}\, du$$

where ν = real quantity.

(D.2) $$I(\beta, \nu) = \sqrt{\frac{\pi}{\beta}}\ (1 - 2\beta \nu^2\ \varphi(\beta, \nu)\ e^{-\beta \nu^2}).$$

A cursory examination of this integral indicates that

(D.3) $$\lim_{\nu \to 0} I(\nu) = \sqrt{\frac{\pi}{\beta}}$$

and

(D.4) $$\lim_{\nu \to \infty} I(\nu) = 0.$$

The first result indicates that

(D.5) $$\lim_{\nu \to 0} (2\beta \nu^2 \varphi\, e^{-\beta \nu^2}) = 0$$

whereas the second means that

(D.6) $$\lim_{\nu \to \infty} (2\beta \nu^2 \varphi\, e^{-\beta \nu^2}) = 1$$

These results are important when examining the nature of $A_k(\nu)$. The above results can be independently obtained by applying l'Hopital's Rule and the fact that

$$\text{(D.7)} \quad \frac{d}{d\nu}(\nu\varphi) = e^{\beta\nu^2}$$

to the function $\Psi(\beta,\nu) \equiv 2\beta\nu^2\varphi e^{-\beta\nu^2}$.

A fairly complete discription can be obtained by considering its derivative for the range $0 \leq \nu \leq +\infty$. Remembering that $\frac{d}{d\nu}(\nu\varphi) = e^{\beta\nu^2}$, it may readily be shown that

$$\text{(D.8)} \quad \frac{d\Psi}{d\nu} = 2\beta\nu\left[1+(1-2\beta\nu^2)\varphi e^{-\beta\nu^2}\right].$$

We see that

$$\text{(D.9)} \quad \lim_{\nu\to 0}\left\{\frac{d\Psi}{d\nu}\right\} = 0.$$

The limit as $\nu \longrightarrow \infty$ is somewhat more difficult to obtain; a complete analysis requires the use of l'Hopital's rule several times. The following somewhat intuitive method, however, yields the correct results. Consider

$$\text{(D.10)} \quad \frac{d\Psi}{d\nu} = 2\beta\nu\left[1+\frac{\varphi}{e^{\beta\nu^2}} - 2\beta\nu^2\varphi e^{-\beta\nu^2}\right].$$

Remembering that limit of $2\beta\nu^2\varphi e^{-\beta\nu^2}$ as $\nu \longrightarrow \infty$ is 1, we have

$$\text{(D.11)} \quad \lim_{\to\infty} \frac{d\Psi}{d\nu} = \lim_{\nu\to\infty} \left\{ 2\beta\nu \frac{\varphi}{e^{\beta\nu^2}} \right\} = \lim_{\nu\to\infty} \left\{ \frac{2\beta e^{\beta\nu^2}}{2\beta\nu\, e^{\beta\nu^2}} \right\}$$

$$= 0.$$

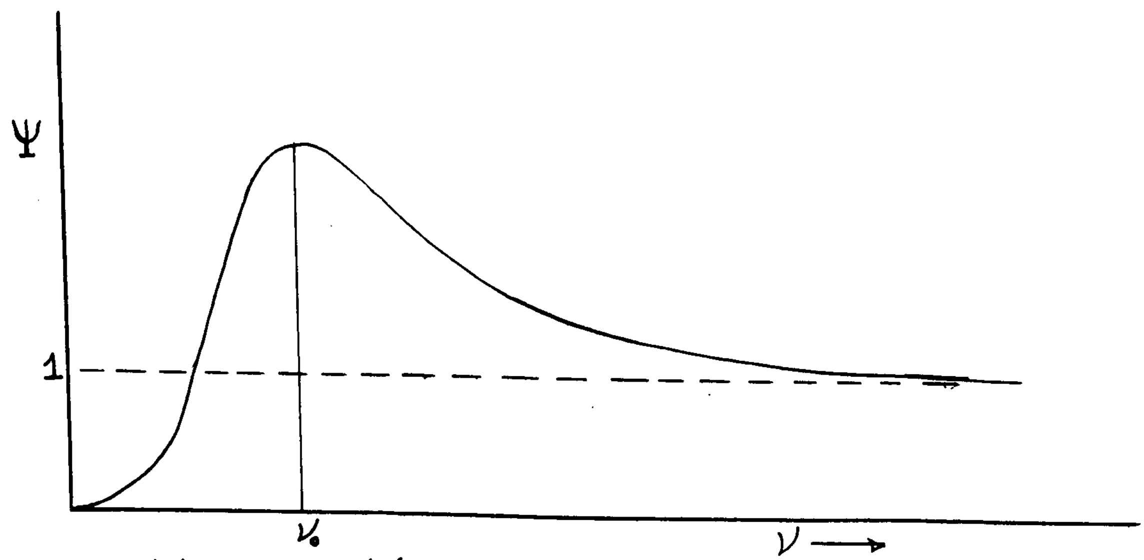
Finally, the third zero in the derivative is given by

$$\text{(D.12)} \quad 2\beta\nu_o^2 \varphi e^{-\beta\nu_o^2} = 1 + \frac{\varphi(\beta, \nu_o^2)}{e^{\beta\nu_o^2}}$$

where

$$\text{(D.13)} \quad 2\beta\nu_o^2 \varphi e^{-\beta\nu_o^2} > 1.$$

Consequently, a typical curve for $\Psi(\beta, \nu)$ would be



Naturally, $\Psi(\beta, \nu) = \Psi(\beta, -\nu)$ and the curve is symmetric about the Ψ axis.

Our function

$$\text{(D.14)} \quad P \int \frac{e^{-\beta u^2}}{(u-\nu)} du = -2\beta \sqrt{\frac{\pi}{\beta}}\, \nu e^{-\beta \nu^2} \varphi(\beta, \nu)$$

corresponds exactly to the real part of the dispersion function, $\mathrm{Re}Z(\nu)$, for the special case where

ν = real quantity

$\beta = 1$.

APPENDIX E

In this section, we shall pay some attention to the derivation of the roots to the dispersion relation

$$(E.1) \qquad 0 = 1 + \frac{2w_p^2}{k^2} \beta \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{\infty} \frac{u_z e^{-\beta u_z^2}}{(u_z - \nu_\alpha)} du_z .$$

The fact that there exists no real roots to this equation is evident from the following. Equation (E.1) is a result of applying the normalization condition to the particular normal mode solution

$$(E.2) \qquad g_{\nu_\alpha}(u_z) = \frac{w_p^2}{k^2} \frac{F'(u_z)}{(u_z - \nu_\alpha)}$$

where ν_α is real discrete quantity. No principal value sign is necessary since, in this case,

$$(E.3) \qquad \frac{w_p^2}{k^2} F'(\nu_\alpha) = 0 .$$

However for a Maxwell-Boltzmann equilibrium distribution, we have shown that

$$(E.4) \qquad \frac{w_p^2}{k^2} F'(u_z) = -2 \frac{w_p^2}{k^2} \beta \sqrt{\frac{\beta}{\pi}} u_z e^{-\beta u_z^2} .$$

Consequently, if there exists real discrete roots to the above dispersion relation, these would occur at

$$(E.5) \qquad \nu_\alpha = 0$$

and

$$(E.6) \qquad \nu_\alpha = \infty.$$

The first condition implies

$$(E.7) \qquad 0 = 1 + 2 \frac{w_p^2}{k^2} \beta$$

which is impossible since this would fix the value of k whereas the second would mean that

$$0 = 1.$$

Clearly, there cannot exist real discrete roots for the case of a Maxwell-Boltzmann equilibrium distribution. Indeed, it is possible to show by the argument theorem as Shure demonstrated [Shure, 1962] that the above dispersion relation can have no solutions for the case where $\mathrm{Im}(\nu_\alpha) \geq 0$. The only case left is that where $\mathrm{Im}(\nu_\alpha) < 0$; the remaining portion of this section will be devoted to the consideration of this case.

Rewriting the above dispersion relation in terms of the plasma dispersion function, we obtain

$$(E.8) \qquad 0 = 1 + \frac{2w_p^2}{k^2} \beta \sqrt{\frac{\beta}{\pi}} \left[\sqrt{\frac{\pi}{\beta}} + \nu_\alpha \int_{-\infty}^{\infty} \frac{e^{-\beta u^2} du}{(u - \nu_\alpha)} \right]$$

$$= 1 + \frac{2w_p^2}{k^2} \beta \sqrt{\frac{\beta}{\pi}} \left[\sqrt{\frac{\pi}{\beta}} + \sqrt{\pi}\, \nu_\alpha\, Z(\sqrt{\beta}\, \nu_\alpha) \right]$$

where $\text{Im}(\nu_\alpha) < 0$. Extricating roots from this general solution is, however, quite complicated; we can not hope to find a general expression such as

$$\nu_\alpha = f(w_p, k, \beta). \tag{E.9}$$

Such a relation can, however, be found for the limited case where $\text{Re}(\nu_\alpha) \gg \text{Im}(\nu_\alpha)$. The following is an outline leading to this relation.

Letting $\nu_\alpha = (\nu_o - i\varepsilon)$ where $|\nu_o| \gg |\varepsilon|$ and $\varepsilon > 0$ we may rewrite the dispersion relation as

$$0 = 1 + \frac{2w_p^2}{k^2}\,\beta\sqrt{\frac{\beta}{\pi}}\int_{-\infty}^{\infty}\frac{\left[u(u-\nu_o) - i\varepsilon\right]\,e^{-\beta u^2}}{(u-\nu_o)^2 + \varepsilon^2}\,du. \tag{E.10}$$

Assuming that ν_o is so large that the numerator goes to zero as $u \longrightarrow \nu_o$, we may drop ε^2 in the denominator. Approximately, we write

$$0 \simeq 1 + \frac{2w_p^2}{k^2}\,\beta\sqrt{\frac{\beta}{\pi}}\left[\int_{-\infty}^{\infty}\frac{ue^{-\beta u^2}}{(u-\nu_o)}du - i\varepsilon\int\frac{ue^{-\beta u^2}}{(u-\nu_o)^2}du\right]. \tag{E.11}$$

Since ε is so small, we may also write

$$0 \simeq 1 + \frac{2w_p^2}{k^2}\,\beta\sqrt{\frac{\beta}{\pi}}\left[P\int\frac{ue^{-\beta u^2}}{(u-\nu_o)}du - i\pi\,\nu_o e^{-\beta\nu_o^2}\right] \tag{E.12}$$

A comparison indicates that

$$\text{(E.13)} \quad \varepsilon \int_{-\infty}^{\infty} \frac{ue^{-\beta u^2}\,du}{(u - \nu_o)^2} = \pi \nu_o \, e^{-\beta \nu_o^2} .$$

Since the meaningful part of the integration is in the region, $\nu_o \gg u$, we may say that

$$\text{(E.14)} \quad (u - \nu_o)^{-2} = \frac{1}{\nu_o^2}\left[1 + \frac{2u}{\nu_o} + \frac{3u^2}{\nu_o^2} + \ldots \right]$$

Substituting this into the above integral, integrating, and retaining the largest term gives

$$\text{(E.15)} \quad \varepsilon \simeq \beta \, \nu_o^3 \sqrt{\frac{\beta}{\pi}} \; \pi \; \nu_o \, e^{-\beta \nu_o^2} .$$

The real part of ν_α may be found in a similar way. Consider the approximation equation (E.12); for the case $\sqrt{\beta} \; \nu_o \gg 1$, we may drop the imaginary term and rewrite this equation as

$$\text{(E.16)} \quad 0 \simeq 1 - \frac{w_p^2}{k^2} \sqrt{\frac{\beta}{\pi}} \; P \int_{-\infty}^{\infty} \frac{\frac{d}{du}\,(e^{-\beta u^2})\,du}{(u - \nu_o)} .$$

Integrating this by parts and expanding $(u - \nu_o)^{-2}$ as was done for the imaginary case yields

$$\text{(E.17)} \quad 1 \simeq \frac{w_p^2}{k^2} \sqrt{\frac{\beta}{\pi}} \; \frac{1}{\nu_o^2} \int \left(1 + \frac{2u}{\nu_o} + \frac{3u^2}{\nu_o^2} + \ldots \right) e^{-\beta u^2} du .$$

Integrating this and retaining the two largest contributions gives

$$(E.18)\quad 1 \simeq \frac{w_p^2}{k^2}\frac{1}{\nu_o^2}\left[1 + \frac{3}{2\beta\,\nu_o^2}\right]$$

or letting $w_o \equiv k\,\nu_o$, we obtain

$$(E.19)\quad \frac{w_p^2}{w_o^2}\left(1 + \frac{3}{2}\frac{k^2}{\beta w_o^2}\right) \simeq 1.$$

The only meaningful root to the above is

$$(E.20)\quad w_o = w_p\left(1 + \frac{3}{4}\frac{k^2}{\beta w_p^2}\right)$$

where the first two terms of the binomial expansion were retained. This root may be put into its more common form by remembering that

$$(E.21)\quad \beta = \frac{m}{2KT} \quad ; \quad w_p^2 = \frac{4\pi ne^2}{m} \quad .$$

Inserting the above into the expression for w_o gives

$$(E.22)\quad w_o = w_p\left(1 + \frac{3}{2}k^2\lambda_D^2\right)$$

where

$$(E.23)\quad \lambda_D = \sqrt{\frac{KT}{4\pi ne^2}}\,,\ \text{Debye length.}$$

Defining $w_\alpha \equiv k\,\nu_\alpha = (w_o - ik\varepsilon)$, yields for the imaginary part

(E.24) $$k\varepsilon \simeq \sqrt{\frac{\pi}{8}}\; w_p \frac{1}{(k\,\lambda_D)^3}\, e^{\frac{-1}{2(k\,\lambda_D)^2}}$$

where $w_o = k\,\nu_o = w_p$ was used as a first approximation. The above results are in full agreement with those derived by Landau; however, his approach is different. Finally, it must be stated that these results are valid whenever $k\,\lambda_D \ll 1$; this condition is a natural result of the method used to derive the real part of w_α. There it was supposed that the imaginary part of the dispersion relation was very small or that $\beta\,\nu_o^2 \gg 1$. Writing $w_o = k\,\nu_o$ and letting $w_o \simeq w_p$ gives

(E.25) $$\beta w_p^2 \simeq \frac{1}{2k^2\lambda_D^2} \gg 1$$

or that

(E.26) $$\frac{1}{4} \gg k\,\lambda_D.$$

As an order of magnitude limitation, the statement $k\,\lambda_D \ll 1$ suffices.